# SHARP CONDITIONS ON GLOBAL EXISTENCE AND BLOW-UP IN A DEGENERATE TWO-SPECIES AND CROSS-ATTRACTION SYSTEM 

JOSÉ ANTONIO CARRILLO AND KE LIN


#### Abstract

We consider a degenerate chemotaxis model with two-species and two-stimuli in dimension $d \geq 3$ and find two critical curves intersecting at one same point which separate the global existence and blow up of weak solutions to the problem. More precisely, above these curves (i.e. subcritical case), the problem admits a global weak solution obtained by the limits of strong solutions to an approximated system. Based on the second moment of solutions, initial data are constructed to make sure blow up occurs in finite time below these curves (i.e. critical and supercritical cases). In addition, the existence or non-existence of minimizers of free energy functional is discussed on the critical curves and the solutions exist globally in time if the size of initial data is small. We also investigate the crossing point between the critical lines in which a refined criteria in terms of the masses is given again to distinguish the dichotomy between global existence and blow up. We also show that the blow ups is simultaneous for both species.


## 1. Introduction

The interaction motion of two cell populations in breast cancer cell invasion models in $\mathbb{R}^{d}(d \geq 3)$ have been described by the following chemotaxis system with two chemicals and nonlinear diffusion (cf. [20, 30])

$$
\begin{cases}u_{t}=\Delta u^{m_{1}}-\nabla \cdot(u \nabla v), & x \in \mathbb{R}^{d}, t>0,  \tag{1.1}\\ -\Delta v=w, & x \in \mathbb{R}^{d}, t>0, \\ w_{t}=\Delta w^{m_{2}}-\nabla \cdot(w \nabla z), & x \in \mathbb{R}^{d}, t>0, \\ -\Delta z=u, & x \in \mathbb{R}^{d}, t>0, \\ u(x, 0)=u_{0}(x), w(x, 0)=w_{0}(x), & x \in \mathbb{R}^{d},\end{cases}
$$

where $m_{1}, m_{2}>1$ are constants. Here, $u(x, t)$ and $w(x, t)$ denote the density of the macrophages and the tumor cells, $v(x, t)$ and $z(x, t)$ denote the concentration of the chemicals produced by $w(x, t)$ and $u(x, t)$, respectively. For simplicity, the initial data are assumed to satisfy

$$
\begin{align*}
u_{0} & \in L^{1}\left(\mathbb{R}^{d} ;\left(1+|x|^{2}\right) d x\right) \cap L^{\infty}\left(\mathbb{R}^{d}\right), \quad \nabla u_{0}^{m_{1}} \in L^{2}\left(\mathbb{R}^{d}\right) \text { and } u_{0} \geq 0 \\
w_{0} & \in L^{1}\left(\mathbb{R}^{d} ;\left(1+|x|^{2}\right) d x\right) \cap L^{\infty}\left(\mathbb{R}^{d}\right), \nabla w_{0}^{m_{2}} \in L^{2}\left(\mathbb{R}^{d}\right) \text { and } w_{0} \geq 0 . \tag{1.2}
\end{align*}
$$

[^0]Since the solutions to the Poisson equations can be written by the Newtonian potential such as

$$
v(x, t)=\mathcal{K} * w=c_{d} \int_{\mathbb{R}^{d}} \frac{w(y, t)}{|x-y|^{d-2}} d y, \quad z(x, t)=\mathcal{K} * u=c_{d} \int_{\mathbb{R}^{d}} \frac{u(y, t)}{|x-y|^{d-2}} d y
$$

with $\mathcal{K}(x)=\frac{c_{d}}{|x|^{d-2}}$ and $c_{d}$ is the surface area of the sphere $S^{d-1}$ in $\mathbb{R}^{d}$, the original system (1.1) can be regarded as the interaction between two populations

$$
\begin{cases}u_{t}=\Delta u^{m_{1}}-\nabla \cdot(u \nabla \mathcal{K} * w), & x \in \mathbb{R}^{d}, t>0  \tag{1.3}\\ w_{t}=\Delta w^{m_{2}}-\nabla \cdot(w \nabla \mathcal{K} * u), & x \in \mathbb{R}^{d}, t>0 \\ u(x, 0)=u_{0}(x), w(x, 0)=w_{0}(x), & x \in \mathbb{R}^{d}\end{cases}
$$

where it follows that the solutions obey the mass conservation

$$
M_{1}:=\int_{\mathbb{R}^{d}} u(x, t) d x=\int_{\mathbb{R}^{d}} u_{0}(x) d x \text { and } M_{2}:=\int_{\mathbb{R}^{d}} w(x, t) d x=\int_{\mathbb{R}^{d}} w_{0}(x) d x
$$

The associated free energy functional $\mathcal{F}$ for (1.1) or (1.3) is given by

$$
\mathcal{F}[u(t), w(t)]=\frac{1}{m_{1}-1} \int_{\mathbb{R}^{d}} u^{m_{1}} d x+\frac{1}{m_{2}-1} \int_{\mathbb{R}^{d}} w^{m_{2}} d x-c_{d} \mathcal{H}[u, w]
$$

which is non-increasing with respect to time since for smooth case it satisfies the following decreasing property

$$
\begin{aligned}
\frac{d}{d t} \mathcal{F}[u(t), w(t)]= & -\int_{\mathbb{R}^{d}} u\left|\frac{m_{1}}{m_{1}-1} \nabla u^{m_{1}-1}-\nabla v\right|^{2} d x \\
& -\int_{\mathbb{R}^{d}} w\left|\frac{m_{2}}{m_{2}-1} \nabla w^{m_{2}-1}-\nabla z\right|^{2} d x
\end{aligned}
$$

where

$$
\mathcal{H}[u, w]=\iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \frac{u(x, t) w(y, t)}{|x-y|^{d-2}} d x d y .
$$

Only one-single population and chemical signal consisting of chemotaxis system is the well-known Keller-Segel model by taking into account volume filling constraints (see [28, 38, 9]) reading as

$$
\begin{cases}u_{t}=\Delta u^{m_{1}}-\nabla \cdot(u \nabla \mathcal{K} * u), & x \in \mathbb{R}^{d}, t>0  \tag{1.4}\\ u(x, 0)=u_{0}(x), & x \in \mathbb{R}^{d}\end{cases}
$$

which has immensely investigated over the last decades. See [3, 23, 28, 39, 13] for the biological motivations and a complete overview of mathematical results for related more general aggregation-diffusion models. Here the diffusion exponent $m_{1}$ is taken to be supercritical $0<m_{1}<m_{c}:=2-2 / d$, critical $m_{1}=m_{c}$ and subcritical $m_{1}>m_{c}$ if $d \geq 3$. The critical number $m_{c}$ is chosen to produces a balance between diffusion and potential drift in mass invariant scaling. For the subcritical $m_{1}>m_{c}$ in the sense that diffusion dominates, the solutions are globally solvable without any restriction on the size of the initial data [29, 43, 45]. However, in the supercritical case, the attraction is stronger leading to a coexistence of global existence of solutions and blow-up behavior. More precisely, finite-time blow up occurs for large initial data, see [11] for $m_{1}=1$, [17] for $m_{1}=2 d /(d+2)$, [16] for $2 d /(d+2)<m_{1}<m_{c}$, and [43] for $1<m_{1}<m_{c}$. But there also exists a global weak solution with decay properties under some smallness condition on
the initial mass [4, 17, 18, 45]. The critical case $m_{1}=m_{c}$ is investigated in [6, 44] showing the existence of a sharp mass constant $M^{*}$ allowing for a dichotomy: if $\|u\|_{1}=M_{1}<M^{*}$ the solutions exist for all time, whereas if $M_{1} \geq M^{*}$ there exists solution with non-positive free energy functional blowing up. In addition, such similar dichotomy was found in [8, 19, 24] earlier in dimension $d=2$ and linear diffusion $m_{1}=1$ for (1.4) with $\mathcal{K}(x)=-1 /(2 \pi) \log |x|$, where $M^{*}$ was replaced by $8 \pi$. We also note that the results in [7] prove that solutions blow up as a delta Dirac at the center of mass as time increases in critical mass $M_{1}=8 \pi$. Sufficient conditions for nonlinear diffusion $m_{1}>1$ to prevent blow up are derived in [9].

The variational viewpoint to analyse problems of the type (1.4) has also been an active field of research. For instance, there have been recent results about the properties of global minimizers of the corresponding free energy functional, including the existence, radial symmetry and uniqueness and so on, since they not only correspond to steady states of (1.4) in some particular cases, but also are candidates for the large time asymptotics of solutions to (1.4). Lion's concentration-compactness principle [36] (see also [2]) can be directly applied to the subcritical $m_{1}>m_{c}$ if $d \geq 3$ and allows the existence of minimizer which further satisfies some regularities properties (see [15]). The uniqueness of minimizer in this case is ensured in [33] and such unique minimizer is also an exponential attractor of solutions of (1.4) when the initial data is radially symmetric and compactly supported by using the mass comparison principle (see [29]). In the critical case $m_{1}=m_{c}$, the free energy functional doses not admit global minimizers except for the critical mass case $M_{1}=M^{*}$ introduced above [10]. Such minimizers were used in [6] to describe the infinite time blow-up profile. For the nonlinear-diffusion in two dimension, the long time asymptotics of solutions is fully characterized in [14] based on the unique existence of radial minimizer of $\mathcal{F}$ [12]. We refer to [5] for a discussion on the existence of many stationary states for $m_{1}=1$ and $d=2$ in the critical case $M_{1}=8 \pi$ and their basins of attraction.

Back to linear two-species system (1.1) in $d=2$, similar to the role of the critical mass $8 \pi$ in (1.4) ([8, 19]), the critical curve $M_{1} M_{2}-4 \pi\left(M_{1}+M_{2}\right)=0$ for two species is discovered in [22]: solutions exist globally if $M_{1} M_{2}-4 \pi\left(M_{1}+M_{2}\right)<0$ and blow up occurs if $M_{1} M_{2}-4 \pi\left(M_{1}+M_{2}\right)>0$. The key tool for the proof of the global existence part is using the Moser-Trudinger inequality as in [42] in two dimensions. One can use partial results in [42] to check that mimimizers indeed exist in the case $M_{1} M_{2}-4 \pi\left(M_{1}+M_{2}\right)=0$. We also mention that such nonlinear system (1.1) and the one-single population system (1.4) can be formally regarded as gradient flows of the free energy functional in the probability measure space with the Euclidean Wasserstein metric [1, 25]. For general $n$-component multipopulations chemotaxis system, in [26, 27] the authors have made considerable progress on these aspects and obtain the global arguments in subcritical and critical cases. The Neumann initial-boundary value problem is analysed in [34, 35, 47, 48].

The aim of this paper is to give a thorough understanding of the well-posedness and asymptotic behavior for (1.1) and (1.3) in $d \geq 3$ and to show the existence or non-existence of global minimizers in critical cases. We make use of bold faces $\mathbf{m}, \mathbf{A}, \mathbf{B}, \mathbf{I}, \mathbf{M}, \cdots$ to denote two-dimensional vectors through the paper and assume that $\mathbf{A}=\left(a_{1}, a_{2}\right) \leq(\geq) \mathbf{B}=\left(b_{1}, b_{2}\right)$ means that $a_{1} \leq(\geq) b_{1}$ and $b_{1} \leq(\geq) b_{2}$, respectively. If $(u, w)$ is a solution of (1.3), then for any $\lambda>0$ the following scaling

$$
u_{\lambda}(x, t)=\lambda^{m_{2}} u\left(\lambda^{\frac{m_{1}+m_{2}-m_{1} m_{2}}{2}} x, \lambda^{m_{1}} t\right), w_{\lambda}(x, t)=\lambda^{m_{1}} w\left(\lambda^{\frac{m_{1}+m_{2}-m_{1} m_{2}}{2}} x, \lambda^{m_{2}} t\right)
$$

is also a solution, where the above scaling becomes mass invariant for both $u$ and $w$ if and only if $\mathbf{m}:=\left(m_{1}, m_{2}\right)=\left(m_{c}, m_{c}\right)$. When $\mathbf{m}$ satisfy

$$
\begin{equation*}
m_{1} m_{2}+2 m_{1} / d=m_{1}+m_{2} \tag{1.5}
\end{equation*}
$$

the mass conservation law only holds for $w$, whereas only $u$ preserves $L^{1}$-norm if

$$
\begin{equation*}
m_{1} m_{2}+2 m_{2} / d=m_{1}+m_{2} \tag{1.6}
\end{equation*}
$$

The curves (1.5) and (1.6) can be shown to be the sharp conditions separating the global existence and blow up. Our main result in Theorem 1.3 shows the following dichotomy: above the two red curves in Figure 1, in the sense that $m_{1} m_{2}+2 m_{1} / d>m_{1}+m_{2}$ or $m_{1} m_{2}+2 m_{2} / d>m_{1}+m_{2}$, weak solutions globally exists and blow up occurs below the red curves for certain initial data regardless of their initial masses (see Theorem 1.3). Several results are also obtained at the critical curves (see Theorem 1.4). In addition, both two lines will intersect at the point $\left(m_{c}, m_{c}\right)$. Therefore, we consider the $\left(m_{1}, m_{2}\right) \in(1, \infty)^{2}$ parameter range divided by the following three critical cases (red curve in Figure (1):

Line $L_{1}: m_{1} m_{2}+2 m_{1} / d=m_{1}+m_{2}$ with $m_{1} \in\left(m_{c}, d / 2\right), m_{2} \in\left(1, m_{c}\right) ;$
Line $L_{2}: m_{1} m_{2}+2 m_{2} / d=m_{1}+m_{2}$ with $m_{1} \in\left(1, m_{c}\right), m_{2} \in\left(m_{c}, d / 2\right)$;
The intersection point $\mathbf{I}:=\left(m_{c}, m_{c}\right)$,


Figure 1. Parameter lines determining the critical regimes.
Based on the above discussion, we say that $\mathbf{m}=\left(m_{1}, m_{2}\right)$ is subcritical if

$$
m_{1} m_{2}+2 m_{1} / d>m_{1}+m_{2} \text { or } m_{1} m_{2}+2 m_{2} / d>m_{1}+m_{2}
$$

and $\mathbf{m}=\left(m_{1}, m_{2}\right)$ is supercritical if

$$
m_{1} m_{2}+2 m_{1} / d<m_{1}+m_{2} \text { and } m_{1} m_{2}+2 m_{2} / d<m_{1}+m_{2}
$$

Notice that this corresponds to be above (subcritical) or below (supercritical) the red curves in Figure 1 We also define subsets of $L^{1}\left(\mathbb{R}^{d}\right)$ as

$$
S_{M_{1}}:=\left\{f \geq 0: f \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{m_{1}}\left(\mathbb{R}^{d}\right) \text { and }\|f\|_{1}=M_{1}\right\}
$$

and

$$
S_{M_{2}}:=\left\{g \geq 0: g \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{m_{2}}\left(\mathbb{R}^{d}\right) \text { and }\|g\|_{1}=M_{2}\right\} .
$$

Now the definition of weak solution for (1.1) or (1.3) is give as
Definition 1.1. Let $m_{1}, m_{2}>1, d \geq 3$ and $T>0$. Suppose the initial data $\left(u_{0}, w_{0}\right)$ satisfies some classical regularities (1.2). Then $(u, w)$ of nonnegative functions defined in $\mathbb{R}^{d} \times(0, T)$ is called a weak solution if

$$
\text { i) }(u, w) \in\left(C\left([0, T) ; L^{1}\left(\mathbb{R}^{d}\right)\right) \cap L^{\infty}\left(\mathbb{R}^{d} \times(0, T)\right)\right)^{2}
$$

$$
\left(u^{m_{1}}, w^{m_{2}}\right) \in\left(L^{2}\left(0, T ; H^{1}\left(\mathbb{R}^{d}\right)\right)\right)^{2} ;
$$

ii) $(u, w)$ satisfies

$$
\begin{aligned}
& \int_{0}^{T} \int_{\mathbb{R}^{d}} u \phi_{1 t} d x d t+\int_{\mathbb{R}^{d}} u_{0}(x) \phi_{1}(x, 0) d x=\int_{0}^{T} \int_{\mathbb{R}^{d}}\left(\nabla u^{m_{1}}-u \nabla v\right) \cdot \nabla \phi_{1} d x d t \\
& \int_{0}^{T} \int_{\mathbb{R}^{d}} w \phi_{2 t} d x d t+\int_{\mathbb{R}^{d}} w_{0}(x) \phi_{2}(x, 0) d x=\int_{0}^{T} \int_{\mathbb{R}^{d}}\left(\nabla w^{m_{2}}-w \nabla z\right) \cdot \nabla \phi_{2} d x d t
\end{aligned}
$$

for any test functions $\phi_{1} \in \mathcal{D}\left(\mathbb{R}^{d} \times[0, T)\right)$ and $\phi_{2} \in \mathcal{D}\left(\mathbb{R}^{d} \times[0, T)\right)$ with $v=\mathcal{K} * w$ and $z=\mathcal{K} * u$.

For a given weak solution, we also define:
Definition 1.2. Let $T>0$. Then $(u, w)$ is called a free energy solution with some regular initial data $\left(u_{0}, w_{0}\right)$ on $(0, T)$ if $(u, w)$ is a weak solution and moreover satisfies $\left(u^{\left(2 m_{1}-1\right) / 2}, w^{\left.\left(2 m_{2}-1\right) / 2\right)}\right) \in\left(L^{2}\left(0, T ; H^{1}\left(\mathbb{R}^{d}\right)\right)\right)^{2}$ and

$$
\begin{align*}
\mathcal{F} & {[u(t), w(t)]+\int_{0}^{t} \int_{\mathbb{R}^{d}} u\left|\frac{m_{1}}{m_{1}-1} \nabla u^{m_{1}-1}-\nabla v\right|^{2} d x d s }  \tag{1.7}\\
& +\int_{0}^{t} \int_{\mathbb{R}^{d}} w\left|\frac{m_{2}}{m_{2}-1} \nabla w^{m_{2}-1}-\nabla z\right|^{2} d x d s \leq \mathcal{F}\left[u_{0}, w_{0}\right]
\end{align*}
$$

for all $t \in(0, T)$ with $v=\mathcal{K} * w$ and $z=\mathcal{K} * u$.
Our first main result for (1.1) or (1.3) above or below lines $L_{1}$ and $L_{2}$ is:
Theorem 1.3. Let $m_{1}, m_{2}>1$. Suppose that the initial data $\left(u_{0}, w_{0}\right)$ with $\left\|u_{0}\right\|_{1}=$ $M_{1},\left\|w_{0}\right\|_{1}=M_{2}$ fulfills (1.2). Then
i) If $\mathbf{m}$ is subcritical, there exists a global free energy solution.
ii) If $\mathbf{m}$ is supercritical, then one can construct large initial data ensuring blow up in finite time.

On the lines $L_{1}, L_{2}$ and intersection point $\mathbf{I}$, our second main result is as follows.
Theorem 1.4. Let $m_{1}, m_{2}>1$. Suppose that the initial data $\left(u_{0}, w_{0}\right)$ with $\left\|u_{0}\right\|_{1}=$ $M_{1},\left\|w_{0}\right\|_{1}=M_{2}$ fulfills (1.2). Then
i) If $\mathbf{m}$ is $\mathbf{I}$, then there exists a number $M_{c}>0$ such that if $M_{1} M_{2}<M_{c}^{2}$, solutions globally exist and if $M_{1} M_{2} /\left(M_{1}^{m_{c}}+M_{2}^{m_{c}}\right)>M_{c}^{2 / d} / 2$, there exists a finite time blow-up
solution. Moreover, non-zero global minimizers of $\mathcal{F}$ exist in $S_{M_{1}} \times S_{M_{2}}$ if we are at the crossing point $\mathbf{M}=\left(M_{c}, M_{c}\right)$.
ii) If $\mathbf{m}$ is on $L_{1}$, there exists a number $M_{2 c}>0$ with the following properties: if $M_{2}<M_{2 c}$, solutions globally exist and $\inf _{f \in S_{M_{1}}} \inf _{g \in S_{M_{2}}} \mathcal{F}[f, g]=0$ if $M_{2}=M_{2 c}$, but there exist no non-zero global minimizers of $\mathcal{F}$ in $S_{M_{1}} \times S_{M_{2}}$. In addition, blow-up solution exists if

$$
\frac{\left(\int_{\mathbb{R}^{d}} u_{0}^{m_{1} / m_{2}} d x\right)^{m_{2} / m_{1}}\left(\int_{\mathbb{R}^{d}} w_{0} d x\right)}{\left(\int_{\mathbb{R}^{d}} u_{0}^{m_{1} / m_{2}} d x\right)^{m_{2}}+\left(\int_{\mathbb{R}^{d}} w_{0} d x\right)^{m_{2}}}>N_{0} \text { with some } N_{0}>0
$$

If $\mathbf{m}$ is on $L_{2}$, there exists $M_{1 c}>0$ with the similar properties for $M_{1}$ and blow-up solution exists if

$$
\frac{\left(\int_{\mathbb{R}^{d}} u_{0} d x\right)\left(\int_{\mathbb{R}^{d}} w_{0}^{m_{2} / m_{1}} d x\right)^{m_{1} / m_{2}}}{\left(\int_{\mathbb{R}^{d}} u_{0} d x\right)^{m_{1}}+\left(\int_{\mathbb{R}^{d}} w_{0}^{m_{2} / m_{1}} d x\right)^{m_{1}}}>N_{0}
$$

iii) A simultaneous blow-up phenomenon exists if $\mathbf{m}$ is critical .

We summarize our second main result on the intersection point I, see Figure 2. The blue curve $M_{1} M_{2}=M_{c}^{2}$ intersects with the green curve $M_{1} M_{2} /\left(M_{1}^{m_{c}}+\right.$ $\left.M_{2}^{m_{c}}\right)=M_{c}^{2 / d} / 2$ at the point $\mathbf{J}=\left(M_{c}, M_{c}\right)$. Theorem 1.4 implies that below the curve $M_{1} M_{2}=M_{c}^{2}$ solutions globally exist and above the curve $M_{1} M_{2} /\left(M_{1}^{m_{c}}+\right.$ $\left.M_{2}^{m_{c}}\right)=M_{c}^{2 / d} / 2$ blow up happens.


Figure 2. Parameter lines on intersection point I.

It is an open problem to determine the sharp relation between the masses leading to dichotomy in the intersection point $\mathbf{I}$ and the long time asymptotics on the red curves $L_{1}$ and $L_{2}$ in Figure 1 .

The organization of the paper is as follows: we first construct an approximated system for (1.1) in Section 2, and provide an sufficient condition for global existence of smooth solution and then obtain global weak solution or free energy solution of (1.1) by passing limits upon a prior estimate. Section 3 deals with properties of free energy functional, including the lower and upper bounds, and the existence or non-existence of non-zero minimizers if $\mathbf{m}$ is critical. Finally, we prove that the solutions are globally solvable if $\mathbf{m}$ is subcritical or critical with small initial data in Section 4 and construct blow-up solutions if $\mathbf{m}$ is supercritical or critical with large masses in Section 5.

## 2. Approximated system

As mentioned in the introduction, we first consider an approximated system

$$
\begin{cases}u_{\epsilon t}(x, t)=\Delta\left(u_{\epsilon}+\epsilon\right)^{m_{1}}-\nabla \cdot\left(u_{\epsilon} \nabla v_{\epsilon}\right), & x \in \mathbb{R}^{d}, t>0  \tag{2.1}\\ v_{\epsilon}=\mathcal{K} * w_{\epsilon}, & x \in \mathbb{R}^{d}, t>0 \\ w_{\epsilon t}(x, t)=\Delta\left(w_{\epsilon}+\epsilon\right)^{m_{2}}-\nabla \cdot\left(w_{\epsilon} \nabla z_{\epsilon}\right), & x \in \mathbb{R}^{d}, t>0 \\ z_{\epsilon}=\mathcal{K} * u_{\epsilon}, & x \in \mathbb{R}^{d}, t>0 \\ u_{\epsilon}(x, 0)=u_{0}^{\epsilon}(x) \geq 0, w_{\epsilon}(x, 0)=w_{0}^{\epsilon}(x) \geq 0, & x \in \mathbb{R}^{d}\end{cases}
$$

with $u_{0}^{\epsilon}$ and $w_{0}^{\epsilon}$ being the convolution of $u_{0}$ and $w_{0}$ with a sequence of mollifiers and $\left\|u_{0}^{\epsilon}\right\|_{1}=\|u\|_{1}=M_{1}$ and $\left\|w_{0}^{\epsilon}\right\|_{1}=\|w\|_{1}=M_{2}$. Then the uniform a priori estimate for solutions to (2.1) is given if $m_{1}$ and $m_{2}$ are suitably large, thus global weak solution or even free energy solution exists by letting $\epsilon$ tends to 0 .

By virtue of the local existence of strong solution for only one-single population chemotaxis system (see [43, Proposition 4.1]), one obtains:
Lemma 2.1. Let $m_{1}, m_{2}>1$. Then there exists $T_{\max }^{\epsilon} \in(0, \infty]$ denoting the maximal existence time such that (2.1) has a unique nonnegative strong solution $\left(u_{\epsilon}, w_{\epsilon}\right) \in$ $\left(\mathbb{W}_{p}^{2,1}\left(Q_{T}\right)\right)^{2}$ with some $p>1$, where $Q_{T}=\mathbb{R}^{d} \times(0, T)$ with $T \in\left(0, T_{\text {max }}^{\epsilon}\right]$ and

$$
\mathbb{W}_{p}^{2,1}\left(Q_{T}\right):=\left\{u \in L^{p}\left(0, T ; W^{2, p}\left(\mathbb{R}^{d}\right)\right) \cap W^{1, p}\left(0, T ; L^{p}\left(\mathbb{R}^{d}\right)\right)\right\} .
$$

Moreover, if $T_{\max }^{\epsilon}<\infty$, then

$$
\lim _{t \rightarrow T_{\max }^{\epsilon}}\left[\left\|u_{\epsilon}(\cdot, t)\right\|_{\infty}+\left\|w_{\epsilon}(\cdot, t)\right\|_{\infty}\right]=\infty
$$

Now we recall the Hardy-Littlewood-Sobolev (HLS) inequality which we frequently use later (see [31] or [32, Chapter 4]).
Lemma 2.2. Let $0<\lambda<d$, and let the Riesz potential $I_{\lambda}(h)$ of a function $h$ defined by

$$
I_{\lambda}(h)(x)=\frac{1}{|x|^{d-\lambda}} * h=\int_{\mathbb{R}^{d}} \frac{h(y)}{|x-y|^{d-\lambda}} d y, \quad x \in \mathbb{R}^{d} .
$$

Then for $h \in L^{\kappa_{1}}\left(\mathbb{R}^{d}\right)$ and for $\kappa_{1}, \kappa_{2}>1$ with $\frac{1}{\kappa_{2}}=\frac{1}{\kappa_{1}}-\frac{\lambda}{d}$, then there exists a sharp constant $C_{H L S}=C_{H L S}\left(d, \lambda, \kappa_{1}\right)>0$ such that

$$
\left\|I_{\lambda}(h)\right\|_{\kappa_{2}} \leq C_{H L S}\|h\|_{\kappa_{1}} .
$$

An equivalent form of the HLS inequality can be stated that if

$$
\frac{1}{p}+\frac{1}{q}=1+\frac{\lambda}{d}
$$

and $h_{1} \in L^{p}\left(\mathbb{R}^{d}\right), h_{2} \in L^{q}\left(\mathbb{R}^{d}\right)$ with $p, q>1$, then there exists a $C_{H L S}=C_{H L S}(d, \lambda, p)>$ 0 such that

$$
\left|\iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \frac{h_{1}(x) h_{2}(y)}{|x-y|^{d-\lambda}} d x d y\right| \leq C_{H L S}\left\|h_{1}\right\|_{p}\left\|h_{2}\right\|_{q}
$$

Inspired by [46], the global solvability of (2.1) can be achieved based on assumptions on the boundedness for $\left\|u_{\epsilon}\right\|_{m_{1}}$ and $\left\|w_{\epsilon}\right\|_{m_{2}}$ with some large $m_{1}$ and $m_{2}$.
Lemma 2.3. Let $T \in\left(0, T_{\max }^{\epsilon}\right]$. Assume that $\mathbf{m}$ satisfies

$$
\begin{equation*}
m_{1} m_{2}+2 m_{1} m_{2} / d>m_{1}+m_{2} \tag{2.2}
\end{equation*}
$$

Suppose that there exists a constant $C>0$ such that $\left(u_{\epsilon}, w_{\epsilon}\right)$ of (2.1) with initial data $\left(u_{0}^{\epsilon}, w_{0}^{\epsilon}\right)$ being the convolution of $\left(u_{0}, w_{0}\right)$ satisfies

$$
\begin{equation*}
\left\|u_{\epsilon}(t)\right\|_{m_{1}} \leq C \text { and }\left\|w_{\epsilon}(t)\right\|_{m_{2}} \leq C \text { for } t \in(0, T) \tag{2.3}
\end{equation*}
$$

Then there exists a constant $C=C\left(d, m_{1}, m_{2}, u_{0 \epsilon}, w_{0 \epsilon}\right)>0$ such that

$$
\begin{equation*}
\left\|\left(u_{\epsilon}(t), w_{\epsilon}(t)\right)\right\|_{r} \leq C \text { for } r \in[1, \infty) \text { and } t \in(0, T) \tag{2.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\left(v_{\epsilon}(t), z_{\epsilon}(t)\right)\right\|_{r}+\left\|\left(\nabla v_{\epsilon}(t), \nabla z_{\epsilon}(t)\right)\right\|_{r} \leq C \text { for } r \in[1, \infty] \text { and } t \in(0, T) \tag{2.5}
\end{equation*}
$$

Proof. We split the proof into three steps.
Step 1. The choices of $p$ and $q$. There exist $\bar{p}>1, \bar{q}>1, r_{1}>1$ and $r_{2}>1$ such that for some $p>\bar{p}$ and $q>\bar{q}$ one has

$$
p> \begin{cases}m_{1}+1, & \text { if } m_{1} \geq \frac{d}{2}, m_{2} \geq \frac{d}{2}  \tag{2.6}\\ \max \left\{m_{1}+1, \frac{\left(m_{1}-1\right)\left(m_{2}-1\right) d}{d-2 m_{2}}, \frac{m_{1}(d-2)}{2 m_{2}}\right\}, & \text { if } m_{1} \geq \frac{d}{2}, m_{2}<\frac{d}{2} \\ \max \left\{m_{1}+1, \frac{d m_{1}^{2}+d-2 m_{1}}{d-2 m_{1}}, \frac{m_{1}(d-2)}{2 m_{2}}\right\}, & \text { if } m_{1}<\frac{d}{2}, m_{2} \geq \frac{d}{2} \\ \max \left\{m_{1}+1, \frac{d m_{1}^{2}+d-2 m_{1}}{d-2 m_{1}}, \frac{\left(m_{1}-1\right)\left(m_{2}-1\right) d}{d-2 m_{2}}, \frac{m_{1}(d-2)}{2 m_{2}}\right\}, & \text { if } m_{1}<\frac{d}{2}, m_{2}<\frac{d}{2}\end{cases}
$$

$$
\begin{gather*}
\frac{1}{r_{1}}<1-\frac{d-2}{\left(q+m_{2}-1\right) d^{\prime}}  \tag{2.7}\\
\frac{1}{r_{1}}>\max \left\{1-\frac{1}{m_{2}}, \frac{d-2}{d} \cdot \frac{p}{p+m_{1}-1}\right\}  \tag{2.8}\\
\frac{1}{r_{2}}>\frac{d-2}{d} \cdot \frac{1}{p+m_{1}-1},  \tag{2.9}\\
\frac{1}{r_{2}}<\min \left\{\frac{1}{m_{1}}, 1-\frac{d-2}{d} \cdot \frac{q}{q+m_{2}-1}\right\} \tag{2.10}
\end{gather*}
$$

and

$$
\begin{equation*}
\frac{\frac{p}{m_{1}}-\frac{1}{r_{1}}}{1-\frac{d}{2}+\frac{\left(p+m_{1}-1\right) d}{2 m_{1}}}+\frac{\frac{1}{m_{2}}-1+\frac{1}{r_{1}}}{1-\frac{d}{2}+\frac{\left(q+m_{2}-1\right) d}{2 m_{2}}}<\frac{2}{d^{\prime}} \tag{2.11}
\end{equation*}
$$

as well as

$$
\begin{equation*}
\frac{\frac{1}{m_{1}}-\frac{1}{r_{2}}}{1-\frac{d}{2}+\frac{\left(p+m_{1}-1\right) d}{2 m_{1}}}+\frac{\frac{q}{m_{2}}-1+\frac{1}{r_{2}}}{1-\frac{d}{2}+\frac{\left(q+m_{2}-1\right) d}{2 m_{2}}}<\frac{2}{d} . \tag{2.12}
\end{equation*}
$$

Let us first pick $r_{1}>1$ and $r_{2}>1$ fulfilling

$$
\begin{equation*}
r_{1}<\min \left\{\frac{d}{d-2}, \frac{m_{2}}{m_{2}-1}\right\} \tag{2.13}
\end{equation*}
$$

and

$$
\begin{equation*}
r_{2}>m_{1}, \tag{2.14}
\end{equation*}
$$

and let

$$
\begin{equation*}
q:=\frac{m_{2}(p-1)}{m_{1}}+1 \tag{2.15}
\end{equation*}
$$

In (2.15), $p>m_{1}+1$ implies $q>m_{2}+1$. The assertions in (2.6)-(2.7) and (2.9) easily hold by sufficiently large $p \geq \bar{p}$ with some $\bar{p}>1$ and $q \geq \bar{q}$ with some $\bar{q}>1$.

To see the possible choice of $r_{1}$ satisfying (2.7)-(2.8), we first observe that $1-$ $\frac{1}{m_{2}} \geq \frac{d-2}{d} \cdot \frac{p}{p+m_{1}-1}$ is true for any $p>1$ if $m_{2} \geq \frac{d}{2}$, and $\frac{1}{r_{1}}>1-\frac{1}{m_{2}}$ holds by (2.13) as well as $1-\frac{1}{m_{2}}<1-\frac{d-2}{\left(q+m_{2}-1\right) d}$ for any $q>1$. Thus the asserted $r_{1}$ can be actually found. When $m_{2}<\frac{d}{2}$, one has $\frac{1}{r_{1}}>\frac{d-2}{d} \cdot \frac{p}{p+m_{1}-1}>1-\frac{1}{m_{2}}$. The first inequality is guaranteed by (2.13) and the second is due to

$$
\begin{aligned}
\frac{d-2}{d} \cdot \frac{p}{p+m_{1}-1}>1-\frac{1}{m_{2}} & \Longleftrightarrow\left(\frac{1}{m_{2}}-\frac{2}{d}\right) p>\frac{\left(m_{1}-1\right)\left(m_{2}-1\right)}{m_{2}} \\
& \Longleftrightarrow p>\frac{\left(m_{1}-1\right)\left(m_{2}-1\right) d}{d-2 m_{2}}
\end{aligned}
$$

by (2.6) if $m_{2}<\frac{d}{2}$. Moreover, from (2.15) and (2.6), $\frac{d-2}{d} \cdot \frac{p}{p+m_{1}-1}<1-\frac{d-2}{\left(q+m_{2}-1\right) d}$. Therefore, one can also choose $r_{1}>1$ satisfying (2.7)-(2.8) in the case $m_{2}<\frac{d}{2}$.

Similar to the choice of $r_{2}$, if $m_{1} \geq \frac{d}{2}$ then it follows from (2.14) that $\frac{1}{r_{2}}<\frac{1}{m_{1}} \leq$ $1-\frac{d-2}{d} \cdot \frac{q}{q+m_{2}-1}$, in which (2.9)-(2.10) can be satisfied due to $\frac{d-2}{d} \cdot \frac{1}{p+m_{1}-1}<\frac{1}{m_{1}}$. If $m_{1}<\frac{d}{2}$, (2.6) implies $\frac{d-2}{d} \cdot \frac{1}{p+m_{1}-1}<1-\frac{d-2}{d} \cdot \frac{q}{q+m_{2}-1}<\frac{1}{m_{1}}$, and the assertion is true.

Since (2.2) ensures

$$
m_{1} / m_{2}-m_{1}<2 m_{1} / d-1,
$$

then

$$
\begin{aligned}
\frac{\frac{p}{m_{1}}-\frac{1}{r_{1}}}{1-\frac{d}{2}+\frac{\left(p+m_{1}-1\right) d}{2 m_{1}}} & +\frac{\frac{1}{m_{2}}-1+\frac{1}{r_{1}}}{1-\frac{d}{2}+\frac{\left(q+m_{2}-1\right) d}{2 m_{2}}} \\
& =\frac{\frac{p}{m_{1}}-\frac{1}{r_{1}}}{1+\frac{(p-1) d}{2 m_{1}}}+\frac{\frac{1}{m_{2}}-1+\frac{1}{r_{1}}}{1+\frac{(q-1) d}{2 m_{2}}} \\
& =\frac{\frac{p}{m_{1}}-\frac{1}{r_{1}}}{1+\frac{(p-1) d}{2 m_{1}}}+\frac{\frac{1}{m_{2}}-1+\frac{1}{r_{1}}}{1+\frac{(p-1) d}{2 m_{1}}} \\
& =\frac{p+\frac{m_{1}}{m_{2}}-m_{1}}{p+\frac{2 m_{1}}{d}-1} \cdot \frac{2}{d}<\frac{2}{d^{\prime}}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\frac{1}{m_{1}}-\frac{1}{r_{2}}}{1-\frac{d}{2}+\frac{\left(p+m_{1}-1\right) d}{2 m_{1}}} & +\frac{\frac{q}{m_{2}}-1+\frac{1}{r_{2}}}{1-\frac{d}{2}+\frac{\left(q+m_{2}-1\right) d}{2 m_{2}}} \\
& =\frac{\frac{q}{m_{2}}+\frac{1}{m_{1}}-1}{1+\frac{(p-1) d}{2 m_{1}}} \\
& =\frac{p+\frac{m_{1}}{m_{2}}-m_{1}}{p+\frac{2 m_{1}}{d}-1} \cdot \frac{2}{d}<\frac{2}{d}
\end{aligned}
$$

which implies (2.11)- (2.12).
Step 2. Inequalities for both $u$ and $w$. For $p>1$ and $q>1$, we test (2.1) ${ }_{1}$ by $u_{\epsilon}^{p-1}$ and integrate to find that

$$
\begin{aligned}
\frac{1}{p} \frac{d}{d t} \int_{\mathbb{R}^{d}} u_{\epsilon}^{p} d x & =-(p-1) \int_{\mathbb{R}^{d}} u_{\epsilon}^{p-2} \nabla u_{\epsilon} \cdot\left(\nabla\left(u_{\epsilon}+\epsilon\right)^{m_{1}}-u_{\epsilon} \nabla v_{\epsilon}\right) d x \\
& \leq-\frac{4 m_{1}(p-1)}{\left(p+m_{1}-1\right)^{2}} \int_{\mathbb{R}^{d}}\left|\nabla u_{\epsilon}^{\frac{p+m_{1}-1}{2}}\right|^{2} d x-\frac{p-1}{p} \int_{\mathbb{R}^{d}} u_{\epsilon}^{p} \Delta v_{\epsilon} d x \\
& =-\frac{4 m_{1}(p-1)}{\left(p+m_{1}-1\right)^{2}} \int_{\mathbb{R}^{d}}\left|\nabla u_{\epsilon}^{\frac{p+m_{1}-1}{2}}\right|^{2} d x+\frac{p-1}{p} \int_{\mathbb{R}^{d}} u_{\epsilon}^{p} w_{\epsilon} d x
\end{aligned}
$$

with $-\Delta v_{\epsilon}=w_{\epsilon}$, and similarly,

$$
\frac{1}{q} \frac{d}{d t} \int_{\mathbb{R}^{d}} w_{\epsilon}^{q} d x \leq-\frac{4 m_{2}(q-1)}{\left(q+m_{2}-1\right)^{2}} \int_{\mathbb{R}^{d}}\left|\nabla w_{\epsilon}^{\frac{q+m_{2}-1}{2}}\right|^{2} d x+\frac{q-1}{q} \int_{\mathbb{R}^{d}} u_{\epsilon} w_{\epsilon}^{q} d x
$$

holds by multiplying $(2.1)_{3}$ by $w_{\epsilon}^{q-1}$ and $-\Delta z_{\epsilon}=u_{\epsilon}$. Then

$$
\begin{align*}
\frac{1}{p} \frac{d}{d t} \int_{\mathbb{R}^{d}} u_{\epsilon}^{p} d x & +\frac{1}{q} \frac{d}{d t} \int_{\mathbb{R}^{d}} w_{\epsilon}^{q} d x+\frac{4 m_{1}(p-1)}{\left(p+m_{1}-1\right)^{2}} \int_{\mathbb{R}^{d}}\left|\nabla u_{\epsilon}^{\frac{p+m_{1}-1}{2}}\right|^{2} d x \\
& +\frac{4 m_{2}(q-1)}{\left(q+m_{2}-1\right)^{2}} \int_{\mathbb{R}^{d}}\left|\nabla w_{\epsilon}^{\frac{q+m_{2}-1}{2}}\right|^{2} d x  \tag{2.16}\\
& \leq \frac{p-1}{p} \int_{\mathbb{R}^{d}} u_{\epsilon}^{p} w_{\epsilon} d x+\frac{q-1}{q} \int_{\mathbb{R}^{d}} u_{\epsilon} w_{\epsilon}^{q} d x
\end{align*}
$$

where

$$
\begin{equation*}
\int_{\mathbb{R}^{d}} u_{\epsilon}^{p} w_{\epsilon} d x \leq\left(\int_{\mathbb{R}^{d}} u_{\epsilon}^{p r_{1}} d x\right)^{\frac{1}{r_{1}}}\left(\int_{\mathbb{R}^{d}} w_{\epsilon}^{r_{1}^{\prime}} d x\right)^{\frac{1}{r_{1}^{\prime}}} \tag{2.17}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{\mathbb{R}^{d}} u_{\epsilon} w_{\epsilon}^{q} d x \leq\left(\int_{\mathbb{R}^{d}} u_{\epsilon}^{r_{2}} d x\right)^{\frac{1}{r_{2}}}\left(\int_{\mathbb{R}^{d}} w_{\epsilon}^{q r_{2}^{\prime}} d x\right)^{\frac{1}{r_{2}^{\prime}}} \tag{2.18}
\end{equation*}
$$

by Hölder's inequality with $r_{1}, r_{2}>1, r_{1}^{\prime}=\frac{r_{1}}{r_{1}-1}$ and $r_{2}^{\prime}=\frac{r_{2}}{r_{2}-1}$. We begin with estimating the right sides of (2.17)-(2.18) based on the choices of $p, q, r_{1}$ and $r_{2}$ in Step 1. The assumption (2.6) ensures

$$
\begin{equation*}
p r_{1}>m_{1}, \tag{2.19}
\end{equation*}
$$

and

$$
\begin{equation*}
p r_{1}<\frac{\left(p+m_{1}-1\right) d}{d-2} \tag{2.20}
\end{equation*}
$$

by (2.8). Then by a variant of the Gagliardo-Nirenberg inequality (see [45, Lemma 6]),

$$
\begin{equation*}
\|\varphi\|_{k_{2}} \leq C^{\frac{2}{r+m-1}}\|\varphi\|_{k_{1}}^{1-\sigma}\left\|\nabla \varphi^{\frac{r+m-1}{2}}\right\|_{2}^{\frac{2 \sigma}{r+m-1}} \tag{2.21}
\end{equation*}
$$

with $m \geq 1, k_{1} \in[1, r+m-1]$ and $1 \leq k_{1} \leq k_{2} \leq \frac{(r+m-1) d}{d-2}$ with $d \geq 3, \sigma=$ $\frac{r+m-1}{2}\left(\frac{1}{k_{1}}-\frac{1}{k_{2}}\right)\left(\frac{1}{d}-\frac{1}{2}+\frac{r+m-1}{2 k_{1}}\right)^{-1}$, we pick $r=p, m=m_{1}, k_{1}=m_{1}, k_{2}=p r_{1}$ in (2.21) and use (2.19)-(2.20) to find

$$
\left(\int_{\mathbb{R}^{d}} u_{\epsilon}^{p r_{1}} d x\right)^{\frac{1}{r_{1}}}=\left\|u_{\epsilon}\right\|_{p r_{1}}^{p} \leq C\left\|u_{\epsilon}\right\|_{m_{1}}^{p(1-\sigma)}\left\|\nabla u_{\epsilon}^{\frac{p+m_{1}-1}{2}}\right\|_{2}^{\frac{2 \sigma}{p+m_{1}-1}}
$$

with

$$
\sigma=\frac{p+m_{1}-1}{2} \frac{\frac{1}{m_{1}}-\frac{1}{p r_{1}}}{\frac{1}{d}-\frac{1}{2}+\frac{p+m_{1}-1}{2 m_{1}}} \in(0,1),
$$

where invoking (2.3) we further obtain

$$
\left(\int_{\mathbb{R}^{d}} u_{\epsilon}^{p r_{1}} d x\right)^{\frac{1}{r_{1}}} \leq C\left\|\nabla u_{\epsilon}^{\frac{p+m_{1}-1}{2}}\right\|_{2}^{\frac{\frac{p}{m_{1}}-\frac{1}{1_{1}}}{\frac{1}{d}-\frac{1}{2}+\frac{p+m_{1}-1}{2 m_{1}}}} .
$$

Likewise, (2.7)-(2.8) warrants that

$$
m_{2}<r_{1}^{\prime}<\frac{\left(q+m_{2}-1\right) d}{d-2}
$$

which allows one to make use of the Gagliardo-Nirenberg inequality and the upper bound for $\|w\|_{m_{2}}$ in (2.3) to estimate

$$
\left(\int_{\mathbb{R}^{d}} w_{\epsilon}^{r_{1}^{\prime}} d x\right)^{\frac{1}{r_{1}^{\prime}}}=\left\|w_{\epsilon}\right\|_{r_{1}^{\prime}} \leq C\left\|\nabla w_{\epsilon}^{\frac{q+m_{2}-1}{2}}\right\|_{2}^{\frac{\frac{1}{m_{2}}-\frac{1}{r_{1}^{\prime}}}{\frac{1}{d}-\frac{1}{2}+\frac{q+m_{2}-1}{2 m_{2}}}}
$$

Then

$$
\begin{align*}
\left(\int_{\mathbb{R}^{d}} u_{\epsilon}^{p r_{1}} d x\right)^{\frac{1}{r_{1}}} & \left(\int_{\mathbb{R}^{d}} w_{\epsilon}^{r_{1}^{\prime}} d x\right)^{\frac{1}{r_{1}^{\prime}}} \\
& \leq C\left\|\nabla u_{\epsilon}^{\frac{p+m_{1}-1}{2}}\right\|_{2}^{\frac{\frac{p}{m_{1}}-\frac{1}{r_{1}}}{\frac{1}{2}-\frac{1}{2}+\frac{p+m_{1}-1}{2 m_{1}}}} \cdot\left\|\nabla w_{\epsilon}^{\frac{q+m_{2}-1}{2}}\right\|_{2}^{\frac{\frac{1}{m_{2}-\frac{1}{r_{1}}}}{\frac{1}{2}+\frac{1}{2}+\frac{q+m_{2}-1}{2 m_{2}}}}  \tag{2.22}\\
& =C\left\|\nabla u_{\epsilon}^{\frac{p+m_{1}-1}{2}}\right\|_{2}^{\frac{\frac{p}{m_{1}-\frac{1}{2}+\frac{1}{r_{1}}}}{\frac{p+m_{1}-1}{2 m_{1}}}} \cdot\left\|\nabla w_{\epsilon}^{\frac{q+m_{2}-1}{2}}\right\|_{2}^{\frac{\frac{1}{m_{2}-1+\frac{1}{r_{1}}}}{\frac{1-\frac{1}{2}+\frac{q+m_{2}-1}{2 m_{2}}}{2}}} .
\end{align*}
$$

To estimate the right side of (2.18), we use (2.10) and (2.9) to obtain

$$
m_{1}<r_{2}<\frac{\left(p+m_{1}-1\right) d}{d-2}
$$

Then the Gagliardo-Nirenberg inequality implies

$$
\left(\int_{\mathbb{R}^{d}} u_{\epsilon}^{r_{2}} d x\right)^{\frac{1}{r_{2}}} \leq C\left\|\nabla u_{\epsilon}^{\frac{p+m_{1}-1}{2}}\right\|_{2}^{\frac{\frac{1}{m_{1}}-\frac{1}{2}}{\frac{1}{2}-\frac{1}{2}+\frac{p+m_{1}-1}{2 m_{1}}}}
$$

by (2.3). We also obtain

$$
m_{2}<q r_{2}^{\prime}<\frac{\left(q+m_{2}-1\right) d}{d-2}
$$

by (2.10) and (2.15), and choose $r=q, m=m_{2}, k_{1}=m_{2}, k_{2}=q r_{2}^{\prime}$ in (2.21) to see that

$$
\begin{aligned}
\left(\int_{\mathbb{R}^{d}} w_{\epsilon}^{q r_{2}^{\prime}} d x\right)^{\frac{1}{r_{2}^{\prime}}} & =\left\|w_{\epsilon}\right\|_{q r_{2}^{\prime}}^{q} \leq C\left\|w_{\epsilon}\right\|_{m_{2}}^{q(1-\sigma)}\left\|\nabla w_{\epsilon}^{\frac{q+m_{2}-1}{2}}\right\|_{2}^{q \frac{2 \sigma}{q+m_{2}-1}} \\
& \leq C\left\|\nabla w_{\epsilon}^{\frac{q+m_{2}-1}{2}}\right\|_{2}^{\frac{\frac{q}{m_{2}}-\frac{1}{2}+\frac{1}{r_{2}^{\prime}}}{\frac{q+m_{2}-1}{2 m_{2}}}} \\
& =C\left\|\nabla w_{\epsilon}^{\frac{q+m_{2}-1}{2}}\right\|_{2}^{\frac{\frac{q}{m_{2}}-1+\frac{1}{r_{2}}}{\frac{1}{d}+\frac{1}{2}+\frac{m_{2}-1}{2 m_{2}}}}
\end{aligned}
$$

with

$$
\sigma=\frac{q+m_{2}-1}{2} \frac{\frac{1}{m_{2}}-\frac{1}{q r_{2}^{\prime}}}{\frac{1}{d}-\frac{1}{2}+\frac{q+m_{2}-1}{2 m_{2}}}
$$

Then

$$
\begin{aligned}
\left(\int_{\mathbb{R}^{d}} u_{\epsilon}^{r_{2}} d x\right)^{\frac{1}{r_{2}}} & \left(\int_{\mathbb{R}^{d}} w_{\epsilon}^{q r_{2}^{\prime}} d x\right)^{\frac{1}{r_{2}^{\prime}}} \\
& \leq C\left\|\nabla u_{\epsilon}^{\frac{p+m_{1}-1}{2}}\right\|_{2}^{\frac{\frac{1}{m_{1}}-\frac{1}{r_{2}}}{\frac{1}{2}-\frac{1}{2}+\frac{p+m_{1}-1}{2 m_{1}}}}\left\|\nabla w_{\epsilon}^{\frac{q+m_{2}-1}{2}}\right\|_{2}^{\frac{q}{\frac{1}{m_{2}}-1+\frac{1}{2}+\frac{1}{r_{2}}} \frac{q+m_{2}-1}{2 m_{2}}}
\end{aligned},
$$

which combines with (2.16) and (2.22) ensures that

$$
\begin{align*}
\frac{1}{p} \frac{d}{d t} \int_{\mathbb{R}^{d}} u_{\epsilon}^{p} d x & +\frac{1}{q} \frac{d}{d t} \int_{\mathbb{R}^{d}} w_{\epsilon}^{q} d x+\frac{4 m_{1}(p-1)}{\left(p+m_{1}-1\right)^{2}} \int_{\mathbb{R}^{d}}\left|\nabla u_{\epsilon}^{\frac{p+m_{1}-1}{2}}\right|^{2} d x \\
& +\frac{4 m_{2}(q-1)}{\left(q+m_{2}-1\right)^{2}} \int_{\mathbb{R}^{d}}\left|\nabla w_{\epsilon}^{\frac{q+m_{2}-1}{2}}\right|^{2} d x \\
\leq & \frac{p-1}{p}\left(\int_{\mathbb{R}^{d}} u_{\epsilon}^{p r_{1}} d x\right)^{\frac{1}{r_{1}}}\left(\int_{\mathbb{R}^{d}} w_{\epsilon}^{r_{1}^{\prime}} d x\right)^{\frac{1}{r_{1}^{\prime}}} \\
& +\frac{q-1}{q}\left(\int_{\mathbb{R}^{d}} u_{\epsilon}^{r_{2}} d x\right)^{\frac{1}{r_{2}}}\left(\int_{\mathbb{R}^{d}} w_{\epsilon}^{q r_{2}^{\prime}} d x\right)^{\frac{1}{r_{2}^{\prime}}}  \tag{2.23}\\
\leq & C\left\|\nabla u_{\epsilon}^{\frac{p+m_{1}-1}{2}}\right\|_{2}^{\frac{\frac{p}{1}}{\frac{1}{2}-\frac{1}{2}+\frac{1}{r_{1}}+\frac{p+m_{1}-1}{2 m_{1}}}} \cdot\left\|\nabla w_{\epsilon}^{\frac{q+m_{2}-1}{2}}\right\|_{2}^{\frac{\frac{1}{m_{2}}-1+\frac{1}{r_{1}}}{\frac{1}{2}-\frac{1}{2}+\frac{q+m_{2}-1}{2 m_{2}}}} \\
& +C\left\|\nabla u_{\epsilon}^{\frac{p+m_{1}-1}{2}}\right\|_{2}^{\frac{\frac{1}{1_{1}}-\frac{1}{r_{2}}}{\frac{1}{2}+\frac{p+m_{1}-1}{2 m_{1}}}} \cdot\left\|\nabla w_{\epsilon}^{\frac{q+m_{2}-1}{2}}\right\|_{2}^{\frac{\frac{q}{m_{2}}-1+\frac{1}{r_{2}}}{\frac{1}{2}+\frac{q+\frac{1}{2}-1}{2 m_{2}}}} .
\end{align*}
$$

Step 3. Boundedness for $u_{\epsilon}$ and $w_{\epsilon}$ in $L^{p_{-}}$and $L^{q}$-spaces. Let $\gamma_{1}>0, \gamma_{2}>0$ be such that $\gamma_{1}+\gamma_{2}<2$. For $\epsilon>0$, a direct application of Young's inequality implies that

$$
\begin{equation*}
\alpha^{\gamma_{1}} \beta^{\gamma_{2}} \leq \epsilon\left(\alpha^{2}+\beta^{2}\right)+C . \tag{2.24}
\end{equation*}
$$

From Step 1, there exist some $p>\bar{p}$ and $q>\bar{q}$ with some $\bar{p}>1$ and $\bar{q}>1$ such that

$$
\frac{\frac{p}{m_{1}}-\frac{1}{r_{1}}}{\frac{1}{d}-\frac{1}{2}+\frac{p+m_{1}-1}{2 m_{1}}}+\frac{\frac{1}{m_{2}}-1+\frac{1}{r_{1}}}{\frac{1}{d}-\frac{1}{2}+\frac{q+m_{2}-1}{2 m_{2}}}<2
$$

and

$$
\frac{\frac{1}{m_{1}}-\frac{1}{r_{2}}}{\frac{1}{d}-\frac{1}{2}+\frac{p+m_{1}-1}{2 m_{1}}}+\frac{\frac{q}{m_{2}}-1+\frac{1}{r_{2}}}{\frac{1}{d}-\frac{1}{2}+\frac{q+m_{2}-1}{2 m_{2}}}<2,
$$

where

$$
\begin{align*}
\frac{1}{p} \frac{d}{d t} \int_{\mathbb{R}^{d}} u_{\epsilon}^{p} d x & +\frac{1}{q} \frac{d}{d t} \int_{\mathbb{R}^{d}} w_{\epsilon}^{q} d x+\frac{2 m_{1}(p-1)}{\left(p+m_{1}-1\right)^{2}} \int_{\mathbb{R}^{d}}\left|\nabla u_{\epsilon}^{\frac{p+m_{1}-1}{2}}\right|^{2} d x  \tag{2.25}\\
& +\frac{2 m_{2}(q-1)}{\left(q+m_{2}-1\right)^{2}} \int_{\mathbb{R}^{d}}\left|\nabla w_{\epsilon}^{\frac{q+m_{2}-1}{2}}\right|^{2} d x \leq C
\end{align*}
$$

by (2.23)-(2.24). One may invoke the Gagliardo-Nirenberg inequality with $\|u\|_{1}=$ $M_{1}$ and $\|w\|_{1}=M_{2}$ and Young's inequality to obtain

$$
\begin{aligned}
\frac{1}{p} \int_{\mathbb{R}^{d}} u_{\epsilon}^{p} d x=\frac{1}{p}\left\|u_{\epsilon}\right\|_{p}^{p} & \leq C\left\|\nabla u_{\epsilon}^{\frac{p+m_{1}-1}{2}}\right\|_{2^{\frac{1}{d}-\frac{1}{2}+\frac{p+m_{1}-1}{2}}}^{\frac{p-1}{2}} \\
& \leq \frac{2 m_{1}(p-1)}{\left(p+m_{1}-1\right)^{2}} \int_{\mathbb{R}^{d}}\left|\nabla u_{\epsilon}^{\frac{p+m_{1}-1}{2}}\right|^{2} d x+C
\end{aligned}
$$

and

$$
\frac{1}{q} \int_{\mathbb{R}^{d}} w_{\epsilon}^{q} d x \leq \frac{2 m_{2}(q-1)}{\left(q+m_{2}-1\right)^{2}} \int_{\mathbb{R}^{d}}\left|\nabla w_{\epsilon}^{\frac{q+m_{2}-1}{2}}\right|^{2} d x+C
$$

by the fact that

$$
\frac{p-1}{\frac{1}{d}-\frac{1}{2}+\frac{p+m_{1}-1}{2}}<2 \text { and } \frac{q-1}{\frac{1}{d}-\frac{1}{2}+\frac{q+m_{2}-1}{2}}<2
$$

Writing $y(t)=\frac{1}{p} \int_{\mathbb{R}^{d}} u_{\epsilon}^{p} d x+\frac{1}{q} \int_{\mathbb{R}^{d}} w_{\epsilon}^{q} d x$, we obtain from (2.25) that

$$
y^{\prime}(t)+y(t) \leq C \text { for } t \in(0, T)
$$

Then

$$
\left\|u_{\epsilon}(t)\right\|_{p} \leq C \text { and }\left\|w_{\epsilon}(t)\right\|_{q} \leq C \text { for } t \in(0, T)
$$

which implies that (2.4) holds.
Step 4. Improve the regularities of $v$ and $z$. As

$$
v_{\epsilon}=\mathcal{K} * w_{\epsilon}=c_{d} \int_{\mathbb{R}^{d}} \frac{w_{\epsilon}(y)}{|x-y|^{d-2}} d y, z_{\epsilon}=c_{d} \int_{\mathbb{R}^{d}} \frac{u_{\epsilon}(y)}{|x-y|^{d-2}} d y,
$$

an application of the HLS inequality ensures that

$$
\begin{align*}
& \left\|\left|\nabla v_{\epsilon}\right|\right\|_{r} \leq c_{d}(d-2)\left\|I_{1}\left(w_{\epsilon}\right)\right\|_{r} \leq C\left\|w_{\epsilon}\right\|_{d r /(d+r)}  \tag{2.26}\\
& \left\|\left|\nabla z_{\epsilon}\right|\right\|_{r} \leq C\left\|u_{\epsilon}\right\|_{d r /(d+r)}
\end{align*}
$$

Furthermore, observing that the Calderon-Zygmund inequality yields the existence of a constant $C=C(r)>0$ such that

$$
\begin{aligned}
&\left\|\partial_{x_{i}} \partial_{x_{j}} v_{\epsilon}\right\|_{r} \leq C\left\|w_{\epsilon}\right\|_{r} \\
&\left\|\partial_{x_{i}} \partial_{x_{j}} z_{\epsilon}\right\|_{r} \leq C\left\|u_{\epsilon}\right\|_{r}, \quad(1 \leq i, j \leq d)
\end{aligned}
$$

we combine (2.4), (2.26) with the Morrey's inequality to see that

$$
\left\|\left(v_{\epsilon}(t), z_{\epsilon}(t)\right)\right\|_{r}+\left\|\left(\nabla v_{\epsilon}(t), \nabla z_{\epsilon}(t)\right)\right\|_{r} \leq C \text { for } r \in[1, \infty] \text { and } t \in(0, T)
$$

Thus we finish our proof.
Upon the boundedness arguments in Lemma 2.3, we obtain a global weak solution by letting a subsequence of $\epsilon$ approaches to 0 .

Lemma 2.4. Under the same assumption in Lemma 2.3, there exists $C>0$ independent of $\epsilon$ such that the strong solution $\left(u_{\epsilon}, w_{\epsilon}\right)$ of (2.1) satisfies

$$
\begin{equation*}
\left\|\left(u_{\epsilon}(t), w_{\epsilon}(t)\right)\right\|_{\infty} \leq C \text { for all } t \in(0, T) \tag{2.27}
\end{equation*}
$$

Moreover, there exists a global weak solution $(u, w)$ of (1.1) which also satisfies a uniform estimate.

Proof. Relying on Lemma 2.3, we apply the Moser's iteration technique to obtain a priori estimate of solution in $L^{\infty}$. Then this local solution can be extended globally in time from the extensibility criterion in Lemma 2.1, which indeed establishes (2.27), see [45, Proposition 10]. Moreover, from (2.27) there exists ( $u, v, w, z$ ) with
the regularities given in Definition 1.1 such that, up to a subsequence, $\epsilon_{n} \rightarrow 0$,

$$
\begin{aligned}
u_{\epsilon_{n}} & \rightarrow u \text { strongly in } C\left([0, T) ; L_{l o c}^{p}\left(\mathbb{R}^{d}\right)\right) \text { and a.e. in } \mathbb{R}^{d} \times(0, T), \\
\nabla u_{\epsilon_{n}}^{m_{1}} & \rightharpoonup \nabla u^{m_{1}} \text { weakly- } * \text { in } L^{\infty}\left((0, T) ; L^{2}\left(\mathbb{R}^{d}\right)\right), \\
w_{\epsilon_{n}} & \rightarrow w \text { strongly in } C\left([0, T) ; L_{l o c}^{p}\left(\mathbb{R}^{d}\right)\right) \text { and a.e. in } \mathbb{R}^{d} \times(0, T), \\
\nabla w_{\epsilon_{n}}^{m_{2}} & \rightharpoonup \nabla w^{m_{2}} \text { weakly-* in } L^{\infty}\left((0, T) ; L^{2}\left(\mathbb{R}^{d}\right)\right), \\
v_{\epsilon_{n}}(t) & \rightarrow v(t) \text { strongly in } L_{l o c}^{r}\left(\mathbb{R}^{d}\right) \text { and a.e. in }(0, T), \\
\nabla v_{\epsilon_{n}}(t) & \rightarrow \nabla v(t) \text { strongly in } L_{l o c}^{r}\left(\mathbb{R}^{d}\right) \text { and a.e. in }(0, T), \\
\Delta v_{\epsilon_{n}}(t) & \rightharpoonup \Delta v(t) \text { weakly in } L_{l o c}^{r}\left(\mathbb{R}^{d}\right) \text { and a.e. in }(0, T), \\
z_{\epsilon_{n}}(t) & \rightarrow z(t) \text { strongly in } L_{l o c}^{r}\left(\mathbb{R}^{d}\right) \text { and a.e. in }(0, T), \\
\nabla z_{\epsilon_{n}}(t) & \rightarrow \nabla z(t) \text { strongly in } L_{l o c}^{r}\left(\mathbb{R}^{d}\right) \text { and a.e. in }(0, T), \\
\Delta z_{\epsilon_{n}}(t) & \rightharpoonup \Delta z(t) \text { weakly in } L_{l o c}^{r}\left(\mathbb{R}^{d}\right) \text { and a.e. in }(0, T),
\end{aligned}
$$

where $p \in(1, \infty), r \in(1, \infty]$ and $T \in(0, \infty)$. Since the above convergence can be calculated in [44, Section 4], we omit the main proof here. Therefore, we have a global weak solution $(u, v, w, z)$ over $\mathbb{R}^{d} \times(0, T)$ with $T>0$.

The weak solution obtained in Lemma 2.4 is also a free energy solution given in Definition 1.2 The proof comes from [43].

Lemma 2.5. Consider a global weak solution in Lemma 2.4, then it is also a global free energy solution $(u, w)$ of (1.1) given in Definition 1.2

Proof. Define a weight function

$$
\psi(|x|)= \begin{cases}1, & \text { for } 0 \leq|x| \leq 1 \\ 1-2(|x|-1)^{2}, & \text { for } 1<|x| \leq \frac{3}{2} \\ 2(2-|x|)^{2}, & \text { for } \frac{3}{2}<|x|<2 \\ 0, & \text { for }|x| \geq 2\end{cases}
$$

and define $\psi_{l}$ by $\psi_{l}(x):=\psi\left(\frac{|x|}{l}\right)$ for any $x \in \mathbb{R}^{d}$ and $l=1,2,3, \cdots$. Evidently,

$$
\left|\nabla \psi_{l}(x)\right| \leq \frac{C}{l}\left(\psi_{l}(x)\right)^{\frac{1}{2}} \text { and }\left|\Delta \psi_{l}(x)\right| \leq \frac{C}{l^{2}}
$$

is valid with some $C>0$. Denote

$$
\begin{aligned}
\mathcal{F}\left[u_{\epsilon}(t), w_{\epsilon}(t)\right]:= & \frac{1}{m_{1}-1} \int_{\mathbb{R}^{d}}\left(u_{\epsilon}+\epsilon\right)^{m_{1}} \psi_{l}(x) d x+\frac{1}{m_{2}-1} \int_{\mathbb{R}^{d}}\left(w_{\epsilon}+\epsilon\right)^{m_{2}} \psi_{l}(x) d x \\
& -\int_{\mathbb{R}^{d}} u_{\epsilon} v_{\epsilon} d x \\
= & \frac{1}{m_{1}-1} \int_{\mathbb{R}^{d}}\left(u_{\epsilon}+\epsilon\right)^{m_{1}} \psi_{l}(x) d x-\int_{\mathbb{R}^{d}} u_{\epsilon} v_{\epsilon} d x \\
& +\frac{1}{m_{2}-1} \int_{\mathbb{R}^{d}}\left(w_{\epsilon}+\epsilon\right)^{m_{2}} \psi_{l}(x) d x-\int_{\mathbb{R}^{d}} w_{\epsilon} z_{\epsilon} d x \\
& +\int_{\mathbb{R}^{d}} \nabla v_{\epsilon} \cdot \nabla z_{\epsilon} d x .
\end{aligned}
$$

Since

$$
\begin{aligned}
& \frac{1}{m_{1}-1} \frac{d}{d t}\left(u_{\epsilon}+\epsilon\right)^{m_{1}} \psi_{l}-\frac{d}{d t}\left(u_{\epsilon} v_{\epsilon}\right)+u_{\epsilon} v_{\epsilon t} \\
& \quad=\nabla \cdot\left(\nabla\left(u_{\epsilon}+\epsilon\right)^{m_{1}}-u_{\epsilon} \nabla v_{\epsilon}\right) \cdot\left(\frac{m_{1}\left(u_{\epsilon}+\epsilon\right)^{m_{1}-1}}{m_{1}-1} \psi_{l}-v_{\epsilon}\right), \\
& \frac{1}{m_{2}-1} \frac{d}{d t}\left(w_{\epsilon}+\epsilon\right)^{m_{2}} \psi_{l}-\frac{d}{d t}\left(w_{\epsilon} z_{\epsilon}\right)+w_{\epsilon} z_{\epsilon t} \\
& \quad=\nabla \cdot\left(\nabla\left(w_{\epsilon}+\epsilon\right)^{m_{2}}-w_{\epsilon} \nabla z_{\epsilon}\right) \cdot\left(\frac{m_{2}\left(w_{\epsilon}+\epsilon\right)^{m_{2}-1}}{m_{2}-1} \psi_{l}-z_{\epsilon}\right)
\end{aligned}
$$

by testing $(\sqrt{(2.1)})_{1}$ by $\frac{m_{1}\left(u_{\epsilon}+\epsilon\right)^{m_{1}-1}}{m_{1}-1} \psi_{l}-v_{\epsilon}$ and (2.1) $)_{3}$ by $\frac{m_{2}\left(w_{\epsilon}+\epsilon\right)^{m_{2}-1}}{m_{2}-1} \psi_{l}-z_{\epsilon}$, then the derivative of $\mathcal{F}\left[u_{\epsilon}(t), w_{\epsilon}(t)\right]$ with respect to time is

$$
\begin{aligned}
\frac{d}{d t} \mathcal{F}\left[u_{\epsilon}(t), w_{\epsilon}(t)\right]= & \frac{1}{m_{1}-1} \frac{d}{d t} \int_{\mathbb{R}^{d}}\left(u_{\epsilon}+\epsilon\right)^{m_{1}} \psi_{l}(x) d x-\frac{d}{d t} \int_{\mathbb{R}^{d}} u_{\epsilon} v_{\epsilon} d x \\
& +\int_{\mathbb{R}^{d}} u_{\epsilon} v_{\epsilon t} d x+\frac{1}{m_{2}-1} \frac{d}{d t} \int_{\mathbb{R}^{d}}\left(w_{\epsilon}+\epsilon\right)^{m_{2}} \psi_{l}(x) d x \\
& -\frac{d}{d t} \int_{\mathbb{R}^{d}} w_{\epsilon} z_{\epsilon} d x+\int_{\mathbb{R}^{d}} w_{\epsilon} z_{\epsilon t} d x \\
= & -\int_{\mathbb{R}^{d}}\left(\nabla\left(u_{\epsilon}+\epsilon\right)^{m_{1}}-u_{\epsilon} \nabla v_{\epsilon}\right) \cdot \nabla\left(\frac{m_{1}\left(u_{\epsilon}+\epsilon\right)^{m_{1}-1}}{m_{1}-1} \psi_{l}-v_{\epsilon}\right) d x \\
& -\int_{\mathbb{R}^{d}}\left(\nabla\left(w_{\epsilon}+\epsilon\right)^{m_{2}}-w_{\epsilon} \nabla z_{\epsilon}\right) \cdot \nabla\left(\frac{m_{2}\left(w_{\epsilon}+\epsilon\right)^{m_{2}-1}}{m_{2}-1} \psi_{l}-z_{\epsilon}\right) d x
\end{aligned}
$$

which can be written as

$$
\begin{align*}
\frac{d}{d t} \mathcal{F}\left[u_{\epsilon}(t), w_{\epsilon}(t)\right]= & -\int_{\mathbb{R}^{d}}\left[\left(u_{\epsilon}+\epsilon\right) \nabla\left(\frac{m_{1}}{m_{1}-1}\left(u_{\epsilon}+\epsilon\right)^{m_{1}-1}-v_{\epsilon}\right)+\epsilon \nabla v_{\epsilon}\right] \\
& \cdot\left[\nabla\left(\frac{m_{1}}{m_{1}-1}\left(u_{\epsilon}+\epsilon\right)^{m_{1}-1}-v_{\epsilon}\right) \psi_{l}\right. \\
& \left.+\left(\frac{m_{1}}{m_{1}-1}\left(u_{\epsilon}+\epsilon\right)^{m_{1}-1}-v_{\epsilon}\right) \nabla \psi_{l}+\nabla v_{\epsilon}\left(\psi_{l}-1\right)+v_{\epsilon} \nabla \psi_{l}\right] d x \\
& -\int_{\mathbb{R}^{d}}\left[\left(w_{\epsilon}+\epsilon\right) \nabla\left(\frac{m_{2}}{m_{2}-1}\left(w_{\epsilon}+\epsilon\right)^{m_{2}-1}-z_{\epsilon}\right)+\epsilon \nabla z_{\epsilon}\right] \\
& \cdot\left[\nabla\left(\frac{m_{2}}{m_{2}-1}\left(w_{\epsilon}+\epsilon\right)^{m_{2}-1}-z_{\epsilon}\right) \psi_{l}\right. \\
& \left.+\left(\frac{m_{2}}{m_{2}-1}\left(w_{\epsilon}+\epsilon\right)^{m_{2}-1}-z_{\epsilon}\right) \nabla \psi_{l}+\nabla z_{\epsilon}\left(\psi_{l}-1\right)+z_{\epsilon} \nabla \psi_{l}\right] d x \\
= & -\int_{\mathbb{R}^{d}} I_{1} \times J_{1} d x-\int_{\mathbb{R}^{d}} I_{2} \times J_{2} d x . \tag{2.28}
\end{align*}
$$

With $U_{\epsilon}:=\frac{m_{1}}{m_{1}-1}\left(u_{\epsilon}+\epsilon\right)^{m_{1}-1}-v_{\epsilon}$, we expand the term $-\int_{\mathbb{R}^{d}} I_{1} \times J_{1} d x$ to find that

$$
\begin{aligned}
-\int_{\mathbb{R}^{d}} I_{1} \times J_{1} d x= & -\int_{\mathbb{R}^{d}}\left(u_{\epsilon}+\epsilon\right) \psi_{l}\left|\nabla U_{\epsilon}\right|^{2} d x-\int_{\mathbb{R}^{d}}\left(u_{\epsilon}+\epsilon\right)\left(U_{\epsilon}+v_{\epsilon}\right) \nabla U_{\epsilon} \cdot \nabla \psi_{l} d x \\
& -\int_{\mathbb{R}^{d}}\left(u_{\epsilon}+\epsilon\right)\left(\psi_{l}-1\right) \nabla U_{\epsilon} \cdot \nabla v_{\epsilon} d x-\epsilon \int_{\mathbb{R}^{d}} \nabla\left(\psi_{l} U_{\epsilon}\right) \cdot \nabla v_{\epsilon} d x \\
& -\epsilon \int_{\mathbb{R}^{d}} \nabla\left(v_{\epsilon}\left(\psi_{l}-1\right)\right) \cdot \nabla v_{\epsilon} d x
\end{aligned}
$$

where by defining $\Omega_{l}=\left\{x \in \mathbb{R}^{d}: l<|x|<2 l\right\}$, upon using Young's inequality, Hölder's inequality and $(a+b)^{m} \leq 2^{m}\left(a^{m}+b^{m}\right)$ with $a, b>0$ and $m>1$, with any $\eta \in(0,1)$ we deduce from $\left|\nabla \psi_{l}\right| \leq \frac{C}{T}\left(\psi_{l}\right)^{\frac{1}{2}}$ and supp $\left|\nabla \psi_{l}\right|=\bar{\Omega}_{l}$ that

$$
\begin{aligned}
-\int_{\mathbb{R}^{d}}\left(u_{\epsilon}+\epsilon\right)\left(U_{\epsilon}+v_{\epsilon}\right) \nabla U_{\epsilon} \cdot \nabla \psi_{l} d x \leq & \eta \int_{\mathbb{R}^{d}}\left(u_{\epsilon}+\epsilon\right) \psi_{l}\left|\nabla U_{\epsilon}\right|^{2} d x \\
& +\frac{C}{\eta l^{2}}\left(\left\|u_{\epsilon}\right\|_{2 m_{1}-1}^{2 m_{1}-1}+\epsilon^{2 m_{1}-1}\left|\Omega_{l}\right|\right), \\
-\int_{\mathbb{R}^{d}}\left(u_{\epsilon}+\epsilon\right)\left(\psi_{l}-1\right) \nabla U_{\epsilon} \cdot \nabla v_{\epsilon} d x= & \int_{\mathbb{R}^{d}}\left(1-\psi_{l}\right) \nabla\left(u_{\epsilon}+\epsilon\right)^{m_{1}} \cdot \nabla v_{\epsilon} d x \\
& +\int_{\mathbb{R}^{d}}\left(u_{\epsilon}+\epsilon\right)\left(\psi_{l}-1\right)\left|\nabla v_{\epsilon}\right|^{2} d x \\
\leq & \int_{\mathbb{R}^{d}}\left(u_{\epsilon}+\epsilon\right)^{m_{1}} \nabla \psi_{l} \cdot \nabla v_{\epsilon} d x \\
& -\int_{\mathbb{R}^{d}}\left(1-\psi_{l}\right)\left(u_{\epsilon}+\epsilon\right)^{m_{1}} \Delta v_{\epsilon} d x \\
\leq & \int_{\mathbb{R}^{d}}\left(u_{\epsilon}+\epsilon\right)^{m_{1}} w_{\epsilon}\left(1-\psi_{l}\right) d x \\
& +\frac{C}{l} \int_{\mathbb{R}^{d}}\left(u^{m_{1}}+\epsilon^{m_{1}}\right)\left|\nabla v_{\epsilon}\right| d x, \\
-\epsilon \int_{\mathbb{R}^{d}} \nabla\left(\psi_{l} U_{\epsilon}\right) \cdot \nabla v_{\epsilon} d x=-\epsilon \int_{\mathbb{R}^{d}} \psi_{l} U_{\epsilon} w_{\epsilon} d x \leq & \epsilon\left\|w_{\epsilon}\right\|_{L^{1}}\left\|U_{\epsilon}\right\|_{L^{\infty},} \\
-\epsilon \int_{\mathbb{R}^{d}} \nabla\left(v_{\epsilon}\left(\psi_{l}-1\right)\right) \cdot \nabla v_{\epsilon} d x \leq & \epsilon \int_{\mathbb{R}^{d}} w_{\epsilon} v_{\epsilon}\left(1-\psi_{l}\right) d x .
\end{aligned}
$$

The regularities of $\left(u_{\epsilon}, v_{\epsilon}, w_{\epsilon}\right)$ from Lemmas 2.3|2.4] assert that

$$
\begin{aligned}
-\int_{\mathbb{R}^{d}} I_{1} \times J_{1} d x \leq & -(1-\eta) \int_{\mathbb{R}^{d}}\left(u_{\epsilon}+\epsilon\right) \psi_{l}\left|\nabla U_{\epsilon}\right|^{2} d x \\
& +\frac{C}{\eta l^{2}}\left(\left\|u_{\epsilon}(t)\right\|_{2 m_{1}-1}^{2 m_{1}-1}+\epsilon^{2 m_{1}-1}\left|\Omega_{l}\right|\right) \\
& +\int_{\mathbb{R}^{d}}\left(u_{\epsilon}+\epsilon\right)^{m_{1}} w_{\epsilon}\left(1-\psi_{l}\right) d x+\frac{C}{l} \int_{\mathbb{R}^{d}}\left(u_{\epsilon}^{m_{1}}+\epsilon^{m_{1}}\right)\left|\nabla v_{\epsilon}\right| d x \\
& +\epsilon\left\|w_{\epsilon}\right\|_{L^{1}}\left\|U_{\epsilon}\right\|_{L^{\infty}}+\epsilon \int_{\mathbb{R}^{d}} w_{\epsilon} v_{\epsilon}\left(1-\psi_{l}\right) d x \\
\leq & -(1-\eta) \int_{\mathbb{R}^{d}}\left(u_{\epsilon}+\epsilon\right) \psi_{l}\left|\nabla U_{\epsilon}\right|^{2} d x+\frac{C}{\eta l^{2}}\left(1+\epsilon^{2 m_{1}-1}\left|\Omega_{l}\right|\right) \\
& +C \int_{\mathbb{R}^{d}} w_{\epsilon}\left(1-\psi_{l}\right) d x+\frac{C}{l}+\epsilon C .
\end{aligned}
$$

Doing a similar argument for $-\int_{\mathbb{R}^{d}} I_{2} \times J_{2} d x$, and integrating (2.28) over time shows that

$$
\begin{aligned}
\mathcal{F}\left[u_{\epsilon}(t), w_{\epsilon}(t)\right] \leq & \mathcal{F}\left[u_{0 \epsilon}, w_{0 \epsilon}\right] \\
& -(1-\eta) \int_{0}^{t} \int_{\mathbb{R}^{d}}\left(u_{\epsilon}+\epsilon\right) \psi_{l}\left|\frac{m_{1}}{m_{1}-1} \nabla\left(u_{\epsilon}+\epsilon\right)^{m_{1}-1}-\nabla v_{\epsilon}\right|^{2} \\
& -(1-\eta) \int_{0}^{t} \int_{\mathbb{R}^{d}}\left(w_{\epsilon}+\epsilon\right) \psi_{l}\left|\frac{m_{2}}{m_{2}-1} \nabla\left(w_{\epsilon}+\epsilon\right)^{m_{2}-1}-\nabla z_{\epsilon}\right|^{2} \\
& +\frac{C T}{\eta l^{2}}\left(1+\epsilon^{2 m_{1}-1}\left|\Omega_{l}\right|+\epsilon^{2 m_{2}-1}\left|\Omega_{l}\right|\right) \\
& +C \int_{0}^{t} \int_{\mathbb{R}^{d}}\left(u_{\epsilon}+w_{\epsilon}\right)\left(1-\psi_{l}\right)+\frac{C T}{l}+\epsilon C T \text { for } t \in(0, T)
\end{aligned}
$$

where as $\epsilon$ tends to 0 ,

$$
\begin{aligned}
\mathcal{F}[u(t), w(t)] \leq & \mathcal{F}\left[u_{0}, w_{0}\right]-(1-\eta) \int_{0}^{t} \int_{\mathbb{R}^{d}} u \psi_{l}\left|\frac{m_{1}}{m_{1}-1} \nabla u^{m_{1}-1}-\nabla v\right|^{2} \\
& -(1-\eta) \int_{0}^{t} \int_{\mathbb{R}^{d}} w \psi_{l}\left|\frac{m_{2}}{m_{2}-1} \nabla w^{m_{2}-1}-\nabla z\right|^{2} \\
& +C \int_{0}^{t} \int_{\mathbb{R}^{d}}\left(u_{0}+w_{0}\right)\left(1-\psi_{l}\right)+\frac{C T}{\eta l^{2}}+\frac{C T}{l} \text { for } t \in(0, T)
\end{aligned}
$$

by the claimed convergence in Lemma 2.4 and a lower semi-continuity of the free energy dissipation. Finally as $l \rightarrow+\infty$ and $\eta \rightarrow 0$,

$$
\begin{aligned}
\mathcal{F}[u(t), w(t)] \leq & \mathcal{F}\left[u_{0}, w_{0}\right]-\int_{0}^{t} \int_{\mathbb{R}^{d}} u\left|\frac{m_{1}}{m_{1}-1} \nabla u^{m_{1}-1}-\nabla v\right|^{2} \\
& -\int_{0}^{t} \int_{\mathbb{R}^{d}} w\left|\frac{m_{2}}{m_{2}-1} \nabla w^{m_{2}-1}-\nabla z\right|^{2} \text { for } t \in(0, T) .
\end{aligned}
$$

Therefore, $(u, w)$ is a free energy solution by the definition.

## 3. The free energy functional

Now we concentrate on a deeper analysis of the energy functional $\mathcal{F}$ given by

$$
\mathcal{F}[u(t), w(t)]=\frac{1}{m_{1}-1} \int_{\mathbb{R}^{d}} u^{m_{1}} d x+\frac{1}{m_{2}-1} \int_{\mathbb{R}^{d}} w^{m_{2}} d x-c_{d} \mathcal{H}[u, w]
$$

with decay property $\mathcal{F}[u(t), w(t)] \leq \mathcal{F}\left[u_{0}, w_{0}\right]$ for $t \geq 0$, where

$$
\mathcal{H}[u, w]=\iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \frac{u(x) w(y)}{|x-y|^{d-2}} d x d y=\int_{\mathbb{R}^{d}} u(x) I_{2}(w)(x) d x=\int_{\mathbb{R}^{d}} w(y) I_{2}(u)(y) d y
$$

The estimate for $\mathcal{H}$ can be given as follows.
Lemma 3.1. Let $\eta>0$, and let $m_{1}, m_{2}, m>1$. If

$$
\begin{equation*}
m<d / 2 \text { and } m m_{2}+2 m m_{2} / d \geq m+m_{2} \tag{3.1}
\end{equation*}
$$

then for any $f \in L^{m}\left(\mathbb{R}^{d}\right)$ and $g \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{m_{2}}\left(\mathbb{R}^{d}\right)$, there holds

$$
\begin{equation*}
|\mathcal{H}[f, g]| \leq \eta\|f\|_{m}^{m}+C \eta^{-\frac{1}{m-1}}\|g\|_{1}^{\frac{m m_{2}+2 m m_{2} / d-m-m_{2}}{(m-1)\left(m_{2}-1\right)}}\|g\|_{m_{2}}^{\frac{m_{2}-2 m m_{2} / d}{(m-1)\left(m_{2}-1\right)}} . \tag{3.2}
\end{equation*}
$$

Moreover, if

$$
\begin{equation*}
m<d / 2 \text { and } m m_{1}+2 m m_{1} / d \geq m+m_{1} \tag{3.3}
\end{equation*}
$$

then for any $f \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{m_{1}}\left(\mathbb{R}^{d}\right)$ and $g \in L^{m}\left(\mathbb{R}^{d}\right)$, there holds

$$
\begin{equation*}
|\mathcal{H}[f, g]| \leq C \eta^{-\frac{1}{m-1}}\|f\|_{1}^{\frac{m_{1} m+2 m_{1} m / d-m_{1}-m}{\left(m_{1}-1\right)(m-1)}}\|f\|_{m_{1}}^{\frac{m_{1}-2 m_{1} m / d}{\left(m_{1}-1\right)(m-1)}}+\eta\|g\|_{m}^{m} \tag{3.4}
\end{equation*}
$$

Proof. Fixing $m \in(1, d / 2)$, using Hölder's inequality with $\frac{1}{m}+\frac{m-1}{m}=1$ and the HLS inequality with $\lambda=2$ in Lemma 2.2, we find that

$$
\begin{equation*}
\mathcal{H}[f, g]=\int_{\mathbb{R}^{d}} f(x) I_{2}(g)(x) d x \leq\|f\|_{m}\left\|I_{2}(g)\right\|_{\frac{m}{m-1}} \leq C_{\mathrm{HLS}}\|f\|_{m}\|g\|_{\frac{m d}{(d+2) m-d}} \tag{3.5}
\end{equation*}
$$

Since the assumption $m+m_{2} \leq m m_{2}+2 m m_{2} / d$ ensures that

$$
1<\frac{m d}{(d+2) m-d} \leq m_{2}
$$

then if $g \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{m_{2}}\left(\mathbb{R}^{d}\right)$ with $m_{2}>1$, the following interpolation inequality holds:

$$
\|g\|_{\frac{m d}{(d+2) m-d}} \leq\|g\|_{1}^{\theta_{1}}\|g\|_{m_{2}}^{1-\theta_{1}}
$$

with $\frac{(d+2) m-d}{m d}=\theta_{1}+\frac{1-\theta_{1}}{m_{2}}, \theta_{1} \in(0,1)$. Hence

$$
\begin{aligned}
|\mathcal{H}[f, g]| & \leq C_{\mathrm{HLS}}\|f\|_{m}\|g\|_{1}^{\theta_{1}}\|g\|_{m_{2}}^{1-\theta_{1}} \\
& \leq \eta\|f\|_{m}^{m}+C \eta^{-\frac{1}{m-1}}\|g\|_{1}^{\frac{m m_{2}+2 m m_{2} / d-m-m_{2}}{(m-1)\left(m_{2}-1\right)}}\|g\|_{m_{2}}^{\frac{m_{2}-2 m m_{2} / d}{(m-1)\left(m_{2}-1\right)}}
\end{aligned}
$$

by Young's inequality, which implies (3.2). (3.4) can be also proved if (3.3) holds.

We establish several variants to the HLS inequality on the lines $L_{1}, L_{2}$ and the intersection point $\mathbf{I}$.
Lemma 3.2. Let $\mathbf{m}$ be on $L_{1}$, and let $f \in L^{m_{1}}\left(\mathbb{R}^{d}\right)$ and $g \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{m_{2}}\left(\mathbb{R}^{d}\right)$. Then

$$
C_{*}:=\sup _{f \neq 0, g \neq 0}\left\{\frac{|\mathcal{H}[f, g]|}{\|f\|_{m_{1}}\|g\|_{1}^{2 / d}\|g\|_{m_{2}}^{1-2 / d}}\right\}<\infty
$$

If $\mathbf{m}$ is on $L_{2}$, and $f \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{m_{1}}\left(\mathbb{R}^{d}\right)$ and $g \in L^{m_{2}}\left(\mathbb{R}^{d}\right)$, then

$$
C_{\star}:=\sup _{f \neq 0, g \neq 0}\left\{\frac{|\mathcal{H}[f, g]|}{\|f\|_{1}^{2 / d}\|f\|_{m_{1}}^{1-2 / d}\|g\|_{m_{2}}}\right\}<\infty
$$

In addtion, assume that $\mathbf{m}$ is $\mathbf{I}$ and $(f, g) \in\left(L^{1}\left(\mathbb{R}^{d}\right) \cap L^{m_{c}}\left(\mathbb{R}^{d}\right)\right)^{2}$. Then

$$
\begin{equation*}
C_{c}:=\sup _{f \neq 0, g \neq 0}\left\{\frac{\mathcal{H}[f, g]}{\|f\|_{1}^{1 / d}\|f\|_{m_{c}}^{m_{c} / 2}\|g\|_{1}^{1 / d}\|g\|_{m_{c}}^{m_{c} / 2}}\right\}<\infty \tag{3.6}
\end{equation*}
$$

Proof. If $\mathbf{m}$ is on $L_{1}$, then $m_{1} \in\left(m_{c}, d / 2\right)$ and using (3.5) with $m=m_{1}$ we have

$$
|\mathcal{H}[f, g]| \leq C_{\mathrm{HLS}}\|f\|_{m_{1}}\|g\|_{\frac{m_{1} d}{(d+2) m_{1}-d}} \leq C_{\mathrm{HLS}}\|f\|_{m_{1}}\|g\|_{1}^{\frac{2}{d}}\|g\|_{m_{2}}^{1-\frac{2}{d}}
$$

Therefore, $C_{*}$ is finite and bounded above by $C_{\text {HLS }}$. It is also easy to see that $C_{\star}$ is controlled by $C_{H L S}$ if $\mathbf{m}$ is on $L_{2}$. Finally, with the help of the HLS inequality and Hölder's inequality, we find that

$$
|\mathcal{H}[f, g]| \leq C_{\mathrm{HLS}}\|f\|_{\frac{2 d}{d+2}}\|g\|_{\frac{2 d}{d+2}} \leq C_{\mathrm{HLS}}\|f\|_{1}^{1 / d}\|f\|_{m_{c}}^{m_{c} / 2}\|g\|_{1}^{1 / d}\|g\|_{m_{c}}^{m_{c} / 2}
$$

if $\mathbf{m}$ is $\mathbf{I}$. Then the definition of $C_{c}$ is valid.
Define

$$
\begin{aligned}
& M_{1 c}=\left(c_{d} C_{\star}\right)^{-d / 2}\left(m_{2} /\left(m_{2}-1\right)\right)^{d / 2}\left(m_{1}-1\right)^{-d\left(m_{2}-1\right) /\left(2 m_{2}\right)}, \\
& M_{2 c}=\left(c_{d} C_{*}\right)^{-d / 2}\left(m_{1} /\left(m_{1}-1\right)\right)^{d / 2}\left(m_{2}-1\right)^{-d\left(m_{1}-1\right) /\left(2 m_{1}\right)}
\end{aligned}
$$

and

$$
M_{c}=\left(2 /\left[c_{d} C_{c}\left(m_{c}-1\right)\right]\right)^{d / 2} .
$$

The lower and upper bounds for $\mathcal{F}$ in the sets $S_{M_{1}} \times S_{M_{2}}$ below is given next.
Lemma 3.3. Let $(f, g)$ satisfy $f \in S_{M_{1}}$ and $g \in S_{M_{2}}$. If $\mathbf{m}$ is on $L_{1}$, then

$$
\begin{align*}
& \left(c_{d} C_{*}\right)^{\frac{m_{1}}{m_{1}-1}}\left(m_{1}-1\right)^{\frac{m_{1}}{m_{1}-1}} m_{1}^{-\frac{m_{1}}{m_{1}-1}}\left(M_{2 c}^{\frac{2 m_{1}}{d\left(m_{1}-1\right)}}-M_{2}^{\frac{2 m_{1}}{d\left(m_{1}-1\right)}}\right)\|g\|_{m_{2}}^{m_{2}} \\
& \leq \mathcal{F}[f, g] \leq \frac{2}{m_{1}-1}\|f\|_{m_{1}}^{m_{1}}  \tag{3.7}\\
& +\left(c_{d} C_{*}\right)^{\frac{m_{1}}{m_{1}-1}}\left(m_{1}-1\right)^{\frac{m_{1}}{m_{1}-1}} m_{1}^{-\frac{m_{1}}{m_{1}-1}}\left(M_{2 c}^{\frac{2 m_{1}}{d\left(m_{1}-1\right)}}+M_{2}^{\frac{2 m_{1}}{d\left(m_{1}-1\right)}}\right)\|g\|_{m_{2}}^{m_{2}}
\end{align*}
$$

and

$$
\inf _{f \in S_{M_{1}}} \inf _{g \in S_{M_{2}}} \mathcal{F}[f, g]=0 \text {, if } M_{2} \in\left(0, M_{2 c}\right] \text {. }
$$

If $\mathbf{m}$ is on $L_{2}$, then

$$
\begin{equation*}
\mathcal{F}[f, g] \geq\left(c_{d} C_{\star}\right)^{\frac{m_{2}}{m_{2}-1}}\left(m_{2}-1\right)^{\frac{m_{2}}{m_{2}-1}} m_{2}^{-\frac{m_{2}}{m_{2}-1}}\left(M_{1 c}^{\frac{2 m_{2}}{d\left(m_{2}-1\right)}}-M_{1}^{\frac{2 m_{2}}{d\left(m_{2}-1\right)}}\right)\|f\|_{m_{1}}^{m_{1}} \tag{3.8}
\end{equation*}
$$

and

$$
\begin{equation*}
\inf _{f \in S_{M_{1}}} \inf _{g \in S_{M_{2}}} \mathcal{F}[f, g]=0, \text { if } M_{1} \in\left(0, M_{1 c}\right] . \tag{3.9}
\end{equation*}
$$

If $\mathbf{m}$ is $\mathbf{I}$, then

$$
\mathcal{F}[f, g] \geq \frac{\left(c_{d} C_{c}\right)^{2}\left(m_{c}-1\right)}{4}\left(M_{c}^{\frac{4}{d}}-M_{1}^{\frac{2}{d}} M_{2}^{\frac{2}{d}}\right)\|g\|_{m_{c}}^{m_{c}}
$$

or

$$
\mathcal{F}[f, g] \geq \frac{\left(c_{d} C_{c}\right)^{2}\left(m_{c}-1\right)}{4}\left(M_{c}^{\frac{4}{d}}-M_{1}^{\frac{2}{d}} M_{2}^{\frac{2}{d}}\right)\|f\|_{m_{c}}^{m_{c}}
$$

Furthermore,

$$
\begin{equation*}
\inf _{f \in S_{M_{1}}} \inf _{g \in S_{M_{2}}} \mathcal{F}[f, g]=0 \text {, if } M_{1} M_{2} \in\left(0, M_{c}^{2}\right] . \tag{3.10}
\end{equation*}
$$

Proof. By Lemma 3.2, $\mathcal{H}$ satisfies

$$
\begin{aligned}
|\mathcal{H}[f, g]| \leq & C_{*}\|f\|_{m_{1}}\|g\|_{1}^{\frac{2}{d}}\|g\|_{m_{2}}^{1-\frac{2}{d}} \\
\leq & \frac{1}{c_{d}\left(m_{1}-1\right)}\|f\|_{m_{1}}^{m_{1}} \\
& +C_{*}\left(c_{d} C_{*}\right)^{\frac{1}{m_{1}-1}}\left(\frac{m_{1}-1}{m_{1}}\right)^{\frac{m_{1}}{m_{1}-1}}\|g\|_{1}^{\frac{2 m_{1}}{d\left(m_{1}-1\right)}}\|g\|_{m_{2}}^{\left(1-\frac{2}{d}\right) \frac{m_{1}}{m_{1}-1}} \\
= & \frac{1}{c_{d}\left(m_{1}-1\right)}\|f\|_{m_{1}}^{m_{1}} \\
& +C_{*}\left(c_{d} C_{*}\right)^{\frac{1}{m_{1}-1}}\left(\frac{m_{1}-1}{m_{1}}\right)^{\frac{m_{1}}{m_{1}-1}}\|g\|_{1}^{\frac{2 m_{1}}{d\left(m_{1}-1\right)}}\|g\|_{m_{2}}^{m_{2}}
\end{aligned}
$$

Then $\mathcal{F}$ can be estimated as

$$
\begin{aligned}
\mathcal{F}[f, g]= & \frac{1}{m_{1}-1}\|f\|_{m_{1}}^{m_{1}}+\frac{1}{m_{2}-1}\|g\|_{m_{2}}^{m_{2}}-c_{d} \mathcal{H}[f, g] \\
\geq & \frac{1}{m_{2}-1}\|g\|_{m_{2}}^{m_{2}} \\
& -\left(c_{d} C_{*}\right)^{\frac{m_{1}}{m_{1}-1}}\left(\frac{m_{1}-1}{m_{1}}\right)^{\frac{m_{1}}{m_{1}-1}}\|g\|_{1}^{\frac{2 m_{1}}{d\left(m_{1}-1\right)}}\|g\|_{m_{2}}^{m_{2}} \\
= & \left(c_{d} C_{*}\right)^{\frac{m_{1}}{m_{1}-1}}\left(\frac{m_{1}-1}{m_{1}}\right)^{\frac{m_{1}}{m_{1}-1}}\left(M_{2 c}^{\frac{2 m_{1}}{d\left(m_{1}-1\right)}}-M_{2}^{\frac{2 m_{1}}{d\left(m_{1}-1\right)}}\right)\|g\|_{m_{2}}^{m_{2}}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{F}[f, g] \leq & \frac{2}{m_{1}-1}\|f\|_{m_{1}}^{m_{1}} \\
& +\left(c_{d} C_{*}\right)^{\frac{m_{1}}{m_{1}-1}}\left(\frac{m_{1}-1}{m_{1}}\right)^{\frac{m_{1}}{m_{1}-1}}\left(M_{2 c}^{\frac{2 m_{1}}{d\left(m_{1}-1\right)}}+M_{2}^{\frac{2 m_{1}}{d\left(m_{1}-1\right)}}\right)\|g\|_{m_{2}}^{m_{2}}
\end{aligned}
$$

In the case $M_{2} \leq M_{2 c}$, since $\mathcal{F} \geq 0$, then the infimum is nonnegative. Taking

$$
h_{1}(x, t)=\frac{M_{1}}{(4 \pi t)^{\frac{d}{2}}} e^{-\frac{|x|^{2}}{4 t}} \text { and } h_{2}(x, t)=\frac{M_{2}}{(4 \pi t)^{\frac{d}{2}}} e^{-\frac{|x|^{2}}{4 t}},
$$

it is obvious that $h_{i} \in L^{1}\left(\mathbb{R}^{d}\right)$ with $\left\|h_{i}\right\|_{1}=M_{i}, i=1,2$, satisfy

$$
\left\|h_{i}\right\|_{m_{i}}^{m_{i}}=O\left(t^{-\frac{d\left(m_{i}-1\right)}{2}}\right),
$$

which implies that $h_{i} \in S_{M_{i}}$ and that $\mathcal{F}\left[h_{1}, h_{2}\right]$ tends to 0 as $t \rightarrow \infty$. Therefore,

$$
\inf _{f \in S_{M_{1}}} \inf _{g \in S_{M_{2}}} \mathcal{F}[f, g]=0
$$

If $\mathbf{m}$ is on $L_{2}$, we have (3.8) one more by the HLS inequality and Hölder's inequality, and take $h_{i}$ above to see (3.9).

If $\mathbf{m}$ is $\mathbf{I}$, since

$$
|\mathcal{H}[f, g]| \leq C_{c} M_{1}^{\frac{1}{d}} M_{2}^{\frac{1}{d}}\|f\|_{m_{c}}^{\frac{m_{c}}{2}}\|g\|_{m_{c}}^{\frac{m_{c}}{2}} \leq \frac{1}{c_{d}\left(m_{c}-1\right)}\|f\|_{m_{c}}^{m_{c}}+\frac{M_{1}^{\frac{2}{d}} M_{2}^{\frac{2}{d}}}{c_{d}\left(m_{c}-1\right) M_{c}^{\frac{4}{d}}}\|g\|_{m_{c}}^{m_{c}}
$$

or

$$
|\mathcal{H}[f, g]| \leq \frac{M_{1}^{\frac{2}{d}} M_{2}^{\frac{2}{d}}}{c_{d}\left(m_{c}-1\right) M_{c}^{\frac{4}{d}}}\|f\|_{m_{c}}^{m_{c}}+\frac{1}{c_{d}\left(m_{c}-1\right)}\|g\|_{m_{c}}^{m_{c}}
$$

by Young's inequality, then $\mathcal{F}$ satisfies

$$
\begin{aligned}
\mathcal{F}[f, g] & \geq \frac{1}{m_{c}-1}\|f\|_{m_{c}}^{m_{c}}+\frac{1}{m_{c}-1}\|g\|_{m_{c}}^{m_{c}}-c_{d} C_{c} M_{1}^{\frac{1}{d}} M_{2}^{\frac{1}{d}}\|f\|_{m_{c}}^{\frac{m_{c}}{2}}\|g\|_{m_{c}}^{\frac{m_{c}}{2_{c}}} \\
& \geq \frac{\left(c_{d} C_{c}\right)^{2}\left(m_{c}-1\right)}{4}\left(\frac{4}{\left(c_{d} C_{c}\left(m_{c}-1\right)\right)^{2}}-M_{1}^{\frac{2}{d}} M_{2}^{\frac{2}{d}}\right)\|g\|_{m_{c}}^{m_{c}}
\end{aligned}
$$

or

$$
\mathcal{F}[f, g] \geq \frac{\left(c_{d} C_{c}\right)^{2}\left(m_{c}-1\right)}{4}\left(\frac{4}{\left(c_{d} C_{c}\left(m_{c}-1\right)\right)^{2}}-M_{1}^{\frac{2}{d}} M_{2}^{\frac{2}{d}}\right)\|f\|_{m_{c}}^{m_{c}}
$$

One finally obtains from

$$
\mathcal{F}[f, g] \leq \frac{2}{m_{c}-1}\|f\|_{m_{1}}^{m_{1}}+\frac{\left(c_{d} C_{c}\right)^{2}\left(m_{c}-1\right)}{4}\left(\frac{4}{\left(c_{d} C_{c}\left(m_{c}-1\right)\right)^{2}}+M_{1}^{\frac{2}{d}} M_{2}^{\frac{2}{d}}\right)\|g\|_{m_{c}}^{m_{c}}
$$

that (3.10) is true by taking $f=h_{1}$ and $g=h_{2}$.
The characterization of non-zero minimizers of $\mathcal{F}$ in $S_{M_{1}} \times S_{M_{2}}$ on critical lines and point is the goal in this subsection. If $\mathbf{m}$ is $\mathbf{I}$, the existence of global minimizers is guaranteed in particular situation. The proof is inspired by [6, Proposition 3.5].

Theorem 3.4. Let $\mathbf{m}$ be $\mathbf{I}$. Then there exist a pair of nonnegative, radially symmetric and non-increasing functions $\left(f^{*}, g^{*}\right) \in\left(L^{1}\left(\mathbb{R}^{d}\right) \cap L^{m_{c}}\left(\mathbb{R}^{d}\right)\right)^{2}$ such that

$$
\mathcal{H}\left[f^{*}, g^{*}\right]=C_{c} .
$$

In addition, there exists a minimizer $(f, g) \in S_{M_{1}} \times S_{M_{2}}$ of $\mathcal{F}$ if $M_{1}=M_{2}=M_{c}$, and the minimizer satisfies

$$
f(x)=g(x)= \begin{cases}\frac{1}{R_{0}^{d}}\left[\zeta\left(\frac{x-x_{0}}{R_{0}}\right)\right]^{d /(d-2)}, & \text { if } x \in B\left(x_{0}, R_{0}\right), \\ 0, & \text { if } x \in \mathbb{R}^{d} \backslash B\left(x_{0}, R_{0}\right)\end{cases}
$$

with some $R_{0}>0$ and $x_{0} \in \mathbb{R}^{d}$, where $\zeta$ is the unique positive radial classical solution to the Lane-Emden equation

$$
\begin{cases}-\Delta \zeta=\frac{m_{c}-1}{m_{c}} \zeta^{1 /\left(m_{c}-1\right)}, & x \in B(0,1) \\ \zeta=0, & x \in \partial B(0,1)\end{cases}
$$

Proof. We claim that if $C_{c}$ in (3.6) is obtained by some non-zero $f$ and $g$, then $g=c_{0} f$ with some $c_{0}$. This is easily verified by the positive definite of $|x-y|^{-(d-2)}$, see [32, Theorem 9.8]. In fact, suppose that there exist a pair of maximizing nonnegative functions $(f, g) \in\left(L^{1}\left(\mathbb{R}^{d}\right) \cap L^{m_{c}}\left(\mathbb{R}^{d}\right)\right)^{2}$ such that

$$
\mathcal{H}[f, g]=C_{c}\|f\|_{1}^{\frac{1}{d}}\|f\|_{m_{c}}^{\frac{m_{c}}{2}}\|g\|_{1}^{\frac{1}{d}}\|g\|_{m_{c}}^{\frac{m_{c}}{2}}
$$

Then by [32, Theorem 9.8] and the HLS inequality,

$$
\begin{align*}
\mathcal{H}[f, g] & \leq \sqrt{\mathcal{H}[f, f]} \cdot \sqrt{\mathcal{H}[g, g]}  \tag{3.11}\\
& \leq C_{c}\|f\|_{1}^{\frac{1}{d}}\|f\|_{m_{c}}^{\frac{m_{c}}{2}}\|g\|_{1}^{\frac{1}{d}}\|g\|_{m_{c}}^{\frac{m_{c}}{2_{c}}}
\end{align*}
$$

However, (3.11) is an equality if and only if $g=c_{0} f$ with some constant $c_{0}$.
Note that

$$
\begin{equation*}
C_{c}=\sup _{f \neq 0}\left\{\frac{\mathcal{H}[f, f]}{\|f\|_{1}^{\frac{2}{\lambda}}\|f\|_{m_{c}}^{m_{c}}}, f \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{m_{c}}\left(\mathbb{R}^{d}\right)\right\} . \tag{3.12}
\end{equation*}
$$

The existence of a maximizing nonnegative, radially symmetric and non-increasing $f^{*}$ with $\left\|f^{*}\right\|_{1}=\left\|f^{*}\right\|_{m_{c}}=1$ for (3.12) has been given in [6. Proposition 3.3]. So choosing $g^{*}=c_{0} f^{*}$, then $\mathcal{H}\left[f^{*}, g^{*}\right]=c_{0} C_{c}$ and the first conclusion has been proved with $c_{0}=1$.

To derive minimizers for $\mathcal{F}$ in the situation $M_{1}=M_{2}=M_{c}$, with $f:=M_{c} f^{*}$ and $g:=M_{c} f^{*}$ we have $(f, g) \in S_{M_{1}} \times S_{M_{2}}$ with $\|f\|_{1}=\|f\|_{m_{c}}=M_{c},\|g\|_{1}=$ $\|g\|_{m_{c}}=M_{c}$. After a careful computation we infers that

$$
\mathcal{F}[f, g]=0
$$

by the definition of $M_{c}$ and $(f, g)$ is a non-zero global minimizer of $\mathcal{F}$ in $S_{M_{1}} \times$ $S_{M_{2}}$. The precisely description of the set of minimizers of $\mathcal{F}$ was derived in [6, Proposition 3.5], we omit it here and have proved the second conclusion.

On $L_{1}$, we assert that there is no non-zero minimizer of $\mathcal{F}$ in $S_{M_{1}} \times S_{M_{2}}$ if $M_{2}=$ $M_{2 c}$. The proof includes two steps: the first one is to derive the nonexistence of non-trivial classical solution to a Lane-Emden system (see Lemma 3.5), and the second is to make a contradiction by the achievement of Euler-Largrange equalities which consist of the Lane-Emden system on the assumption that minimizers of its free energy exist (see Theorem 3.6).

Lemma 3.5. Let $M_{1}, M_{2}, \rho>0$, and let $m_{1}>1$ and $m_{2}>1$. Consider a Lane-Emden system

$$
\begin{cases}-\Delta \vartheta(x)=\frac{m_{1}-1}{m_{1}} \varsigma^{\frac{1}{m_{2}-1}}(x), & x \in \Omega_{1}=\mathbb{R}^{d}  \tag{3.13}\\ -\Delta \varsigma(x)=\frac{m_{2}-1}{m_{2}} \vartheta^{\frac{1}{m_{1}-1}}(x), & x \in \Omega_{2}=B(0, \rho) \\ \varsigma(x)=0, \quad x \in \mathbb{R}^{d} \backslash \Omega_{2}\end{cases}
$$

Then (3.13) does not admit any nonnegative and non-trivial classical solution $(\vartheta, \varsigma) \in$ $\left(L^{1 /\left(m_{1}-1\right)}\left(\mathbb{R}^{d}\right) \cap L^{m_{1} /\left(m_{1}-1\right)}\left(\mathbb{R}^{d}\right)\right) \times\left(L^{1 /\left(m_{2}-1\right)}\left(\mathbb{R}^{d}\right) \cap L^{m_{2} /\left(m_{2}-1\right)}\left(\mathbb{R}^{d}\right)\right)$ with $\left\|\vartheta^{1 /\left(m_{1}-1\right)}\right\|_{1}=M_{1}$ and $\left\|\varsigma^{1 /\left(m_{2}-1\right)}\right\|_{1}=M_{2}$, provided that $\mathbf{m}$ is on $L_{1}$.
Proof. Let

$$
q:=\frac{1}{m_{1}-1} \in\left(\frac{2}{d-2}, \frac{d}{d-2}\right) .
$$

The existence/nonexistence of solutions to the general form of Lane-Emden system has been investigated in [37, 40, 41], for example. However, the solvability of (3.13) involving both whole space and bounded domains has not yet known as far
as we know. We assert that there exists no non-trivial classical solution for (3.13) if $\mathbf{m}$ is on $L_{1}$.

Consider the following properties: Suppose that $\omega \in C^{2}\left(\mathbb{R}^{d}\right)$ is non-trivial and satisfies $\Delta w \leq 0, x \in \mathbb{R}^{d}$. Then

$$
\begin{equation*}
\omega(x) \geq C|x|^{2-d}, \quad|x| \geq 1 \tag{3.14}
\end{equation*}
$$

by the strong maximum principle (see [40, Proposition 3.4]). Relying on the finite of $\|\vartheta\|_{q}$, we have the following contradiction: For $R>1$,

$$
M_{1} \geq \int_{B(0, R)} \vartheta^{q}=c_{d} \int_{0}^{R} \int_{S^{d-1}} \vartheta^{q}(r, \theta) r^{d-1} d S(\theta) d r
$$

where one combines with the fact that $\Delta \vartheta \leq 0$ for $x \in \Omega_{1}=\mathbb{R}^{d}$ and (3.14) to see that

$$
\begin{aligned}
M_{1} \geq & \geq \int_{1}^{R} r^{d-1+q(2-d)} d r=C \int_{1}^{R} r^{\frac{d m_{1}+2-2 d}{m_{1}-1}-1} d r \\
& =\frac{C\left(m_{1}-1\right)}{d m_{1}+2-2 d}\left(R^{\frac{d m_{1}+2-2 d}{m_{1}-1}}-1\right) \rightarrow \infty \text { as } R \rightarrow \infty
\end{aligned}
$$

due to $m_{1}>m_{c}=2-2 / d$. So (3.13) has no non-trivial and nonnegative classical solution.

Theorem 3.6. Let $\mathbf{m}$ be on $L_{1}$. For all $M_{2} \leq M_{2 c}$, then $\mathcal{F}$ does not admit any non-zero minimizer in $S_{M_{1}} \times S_{M_{2}}$.

Proof. The left inequality in (3.7) in Lemma 3.3 makes sure that there exists no minimizer if $M_{2}<M_{2 c}$. Thus we only consider $M_{2}=M_{2 c}$ and prove it by contradiction.
Step 1. Necessary conditions for global minimizers of $\mathcal{F}$. We assume that minimizers exist and try to present some basic properties of them. Suppose that $\left(f^{*}, g^{*}\right) \in$ $S_{M_{1}} \times S_{M_{2}}$ is a minimizer of $\mathcal{F}$ in the sense that $\mathcal{F}\left[f^{*}, g^{*}\right]=0$. Then

$$
\begin{align*}
& \frac{1}{m_{1}-1}\left\|f^{*}\right\|_{m_{1}}^{m_{1}}+\frac{1}{m_{2}-1}\left\|g^{*}\right\|_{m_{2}}^{m_{2}}=c_{d} \mathcal{H}\left[f^{*}, g^{*}\right] \\
& \leq c_{d} C_{*}\left\|f^{*}\right\|_{m_{1}}\left\|g^{*}\right\|_{1}^{2 / d}\left\|g^{*}\right\|_{m_{2}}^{1-2 / d} \\
& \leq \frac{1}{m_{1}-1}\left\|f^{*}\right\|_{m_{1}}^{m_{1}}+\left(c_{d} C_{*}\right)^{\frac{m_{1}}{m_{1}-1}}\left(\frac{m_{1}-1}{m_{1}}\right)^{\frac{m_{1}}{m_{1}-1}}\left\|g^{*}\right\|_{1}^{\frac{2 m_{1}}{d\left(m_{1}-1\right)}}\left\|g^{*}\right\|_{m_{2}}^{\left(1-\frac{2}{d}\right) \frac{m_{1}}{m_{1}-1}} \\
& =\frac{1}{m_{1}-1}\left\|f^{*}\right\|_{m_{1}}^{m_{1}}+\frac{1}{m_{2}-1} M_{2 c}^{-\frac{2 m_{1}}{d\left(m_{1}-1\right)}}\left\|g^{*}\right\|_{1}^{\frac{2 m_{1}}{d\left(m_{1}-1\right)}}\left\|g^{*}\right\|_{m_{2}}^{m_{2}}  \tag{3.15}\\
& =\frac{1}{m_{1}-1}\left\|f^{*}\right\|_{m_{1}}^{m_{1}}+\frac{1}{m_{2}-1} M_{2 c}^{-\frac{2 m_{1}}{d\left(m_{1}-1\right)}} M_{2}^{\frac{2 m_{1}}{d\left(m_{1}-1\right)}}\left\|g^{*}\right\|_{m_{2}}^{m_{2}} \\
& =\frac{1}{m_{1}-1}\left\|f^{*}\right\|_{m_{1}}^{m_{1}}+\frac{1}{m_{2}-1}\left\|g^{*}\right\|_{m_{2}}^{m_{2}}
\end{align*}
$$

by the HLS inequality, Young's inequality, the definition of $M_{2 c}$ and $M_{2}=M_{2 c}$. As a consequence of (3.15), we obtain that

$$
\begin{align*}
\left\|f^{*}\right\|_{m_{1}}^{m_{1}} & =\frac{1}{m_{2}-1} M_{2 c}^{-\frac{2 m_{1}}{d\left(m_{1}-1\right)}}\left\|g^{*}\right\|_{1}^{\frac{2 m_{1}}{d\left(m_{1}-1\right)}}\left\|g^{*}\right\|_{m_{2}}^{m_{2}} \\
& =\frac{1}{m_{2}-1} M_{2 c}^{-\frac{2 m_{1}}{d\left(m_{1}-1\right)}} M_{2}^{\frac{2 m_{1}}{d\left(m_{1}-1\right)}}\left\|g^{*}\right\|_{m_{2}}^{m_{2}}  \tag{3.16}\\
& =\frac{1}{m_{2}-1}\left\|g^{*}\right\|_{m_{2}}^{m_{2}}
\end{align*}
$$

and

$$
\mathcal{H}\left[f^{*}, g^{*}\right]=C_{*}\left\|f^{*}\right\|_{m_{1}}\left\|g^{*}\right\|_{1}^{2 / d}\left\|g^{*}\right\|_{m_{2}}^{1-2 / d}=\frac{m_{1}}{c_{d}\left(m_{1}-1\right)\left(m_{2}-1\right)}\left\|g^{*}\right\|_{m_{2}}^{m_{2}}
$$

Step 2. The Euler-Lagrange equalities. Let $f$ and $g$ be symmetric rearrangement of $f^{*}$ and $g^{*}$. Then $(f, g) \in S_{M_{1}} \times S_{M_{2}}$ satisfies

$$
\begin{equation*}
\|f\|_{m_{1}}^{m_{1}}=\left\|f^{*}\right\|_{m_{1}}^{m_{1}}=\frac{1}{m_{2}-1}\left\|g^{*}\right\|_{m_{2}}^{m_{2}}=\frac{1}{m_{2}-1}\|g\|_{m_{2}}^{m_{2}} \tag{3.17}
\end{equation*}
$$

and

$$
\mathcal{H}[f, g] \geq \mathcal{H}\left[f^{*}, g^{*}\right]
$$

by (3.16) and the Riesz rearrangement properties [31, Lemma 2.1]. Obviously, $\mathcal{F}[f, g]=0$ and $(f, g)$ is also a minimizer of $\mathcal{F}$. Note that

$$
\begin{equation*}
c_{d} \mathcal{H}[f, g]=\frac{m_{1}}{m_{1}-1}\|f\|_{m_{1}}^{m_{1}}=\frac{m_{1}}{\left(m_{1}-1\right)\left(m_{2}-1\right)}\|g\|_{m_{2}}^{m_{2}} . \tag{3.18}
\end{equation*}
$$

Given $\Omega_{10}=\left\{x \in \mathbb{R}^{d}: f(x)=0\right\}$ and $\Omega_{1+}=\left\{x \in \mathbb{R}^{d}: f(x)>0\right\}$ and introduce $\phi_{1} \in C_{0}^{\infty}\left(\mathbb{R}^{d}\right)$ with $\phi_{1}(x)=\phi_{1}(-x)$ and

$$
\psi_{1}(x)=\frac{f(x)}{M_{1}}\left(\phi_{1}(x)-\frac{1}{M_{1}} \int_{\mathbb{R}^{d}} f(x) \phi_{1}(x) d x\right)
$$

Then for $f \in S_{M_{1}}$ and fix $\epsilon \in\left(0, \epsilon_{0}:=M_{1}\left(2\left\|\phi_{1}\right\|_{\infty}\right)^{-1}\right)$, there holds

$$
\left\|f+\epsilon \psi_{1}\right\|_{1}=M_{1}
$$

and

$$
\begin{aligned}
f+\epsilon \psi_{1} & =f\left(1+\frac{\epsilon}{M_{1}}\left(\phi_{1}(x)-\frac{1}{M_{1}} \int_{\mathbb{R}^{d}} f(x) \phi_{1}(x) d x\right)\right) \\
& \geq f\left(1-\frac{2\left\|\phi_{1}\right\|_{\infty} \epsilon}{M_{1}}\right) \geq 0
\end{aligned}
$$

which implies that $f+\epsilon \psi_{1} \in S_{M_{1}}$. Moreover, supp $\left(\psi_{1}\right) \subset \bar{\Omega}_{1+}$. Then

$$
\frac{\mathcal{F}\left[f+\epsilon \psi_{1}, g\right]-\mathcal{F}[f, g]}{\epsilon}=\frac{1}{m_{1}-1} \int_{\Omega_{1+}} \frac{\left(f+\epsilon \psi_{1}\right)^{m_{1}}-f^{m_{1}}}{\epsilon}-\int_{\mathbb{R}^{d}} \mathcal{K} * g(x) \psi_{1}(x) d x
$$

According to $\mathcal{F}\left[f+\epsilon \psi_{1}, g\right] \geq \mathcal{F}[f, g]$, as $\epsilon \rightarrow 0$, Lebesgue's dominated convergence theorem shows that

$$
\int_{\mathbb{R}^{d}}\left(\frac{m_{1}}{m_{1}-1} f^{m_{1}-1}(x)-\mathcal{K} * g(x)\right) \psi_{1}(x) d x \geq 0
$$

By replacing $-\psi_{1}$ by $\psi_{1}$, one also obtains from above to see that

$$
\int_{\mathbb{R}^{d}}\left(\frac{m_{1}}{m_{1}-1} f^{m_{1}-1}(x)-\mathcal{K} * g(x)\right) \psi_{1}(x) d x=0
$$

where

$$
\begin{aligned}
0= & \frac{1}{M_{1}} \int_{\mathbb{R}^{d}}\left(\frac{m_{1}}{m_{1}-1} f^{m_{1}-1}(x)-\mathcal{K} * g(x)\right) f(x) \phi_{1}(x) d x \\
& -\frac{1}{M_{1}^{2}} \int_{\mathbb{R}^{d}} f(x) \phi_{1}(x) d x \cdot \int_{\mathbb{R}^{d}}\left(\frac{m_{1}}{m_{1}-1} f^{m_{1}}(x)-\mathcal{K} * f(x) g(x)\right) d x \\
= & \frac{1}{M_{1}} \int_{\mathbb{R}^{d}}\left(\frac{m_{1}}{m_{1}-1} f^{m_{1}-1}(x)-\mathcal{K} * g(x)\right) f(x) \phi_{1}(x) d x
\end{aligned}
$$

by (3.18). For any choice of symmetric test function $\phi_{1} \in C_{0}^{\infty}\left(\mathbb{R}^{d}\right)$, we also obtain

$$
\frac{m_{1}}{m_{1}-1} f^{m_{1}-1}(x)-\mathcal{K} * g(x)=0 \text { a.e. in } \mathbb{R}^{d}
$$

For $g$, arguing similarly as above and we define $\Omega_{20}=\left\{x \in \mathbb{R}^{d}: g(x)=0\right\}$ and $\Omega_{2+}=\left\{x \in \mathbb{R}^{d}: g(x)>0\right\}$ and introduce $\phi_{2} \in C_{0}^{\infty}\left(\mathbb{R}^{d}\right)$ with $\phi_{2}(x)=\phi_{2}(-x)$ and

$$
\psi_{2}(x)=\frac{g(x)}{M_{2}}\left(\phi_{2}(x)-\frac{1}{M_{2}} \int_{\mathbb{R}^{d}} g(x) \phi_{2}(x) d x\right)
$$

Then for $g \in S_{M_{2}}$ and fix $\epsilon \in\left(0, M_{2}\left(2\left\|\phi_{2}\right\|_{\infty}\right)^{-1}\right)$, there holds $g+\epsilon \psi_{2} \in S_{M_{2}}$. Then

$$
\begin{aligned}
\frac{\mathcal{F}\left[f, g+\epsilon \psi_{2}\right]-\mathcal{F}[f, g]}{\epsilon}= & \frac{1}{m_{2}-1} \int_{\Omega_{2+}} \frac{\left(g+\epsilon \psi_{2}\right)^{m_{2}}-g^{m_{2}}}{\epsilon} d y \\
& -\int_{\mathbb{R}^{d}} \mathcal{K} * f(y) \psi_{2}(y) d y
\end{aligned}
$$

where by Lebesgue's dominated convergence theorem again and replacing $-\psi_{2}$ by $\psi_{2}$, it follows that

$$
\int_{\mathbb{R}^{d}}\left(\frac{m_{2}}{m_{2}-1} g^{m_{2}-1}(y)-\mathcal{K} * f(y)\right) \psi_{2}(y) d y=0
$$

Then (3.17) and (3.18) imply that

$$
\begin{aligned}
0= & \frac{1}{M_{2}} \int_{\mathbb{R}^{d}}\left(\frac{m_{2}}{m_{2}-1} g^{m_{2}-1}(y)-\mathcal{K} * f(y)\right) g(y) \phi_{2}(y) d y \\
& -\frac{1}{M_{2}^{2}} \int_{\mathbb{R}^{d}} g(y) \phi_{2}(y) d y \cdot \int_{\mathbb{R}^{d}}\left(\frac{m_{2}}{m_{2}-1} g^{m_{2}}(y)-\mathcal{K} * f(y) g(y)\right) d y \\
= & \frac{1}{M_{2}} \int_{\mathbb{R}^{d}}\left(\frac{m_{2}}{m_{2}-1} g^{m_{2}-1}(y)-\mathcal{K} * f(y)\right) g(y) \phi_{2}(y) d y \\
& +\frac{2 m_{1}}{M_{2}^{2}\left(d-2 m_{1}\right)}\|g\|_{m_{2}}^{m_{2}} \int_{\mathbb{R}^{d}} g(y) \phi_{2}(y) d y \\
= & \frac{1}{M_{2}} \int_{\mathbb{R}^{d}}\left(\frac{m_{2}}{m_{2}-1} g^{m_{2}-1}(y)-\mathcal{K} * f(y)+\frac{2 m_{1}\|g\|_{m_{2}}^{m_{2}}}{M_{2}\left(d-2 m_{1}\right)}\right) g(y) \phi_{2}(y) d y
\end{aligned}
$$

on $L_{1}$. Therefore,

$$
\begin{equation*}
\frac{m_{2}}{m_{2}-1} g^{m_{2}-1}-\mathcal{K} * f+\frac{2 m_{1}}{M_{2}\left(d-2 m_{1}\right)}\|g\|_{m_{2}}^{m_{2}}=0 \text { a.e. in } \bar{\Omega}_{2+} . \tag{3.19}
\end{equation*}
$$

where we extend above equality to the whole space in the sense that

$$
\frac{m_{2}}{m_{2}-1} g^{m_{2}-1}=\left(\mathcal{K} * f-\frac{2 m_{1}}{M_{2}\left(d-2 m_{1}\right)}\|g\|_{m_{2}}^{m_{2}}\right)_{+} \text {a.e. in } \mathbb{R}^{d} .
$$

Since $g$ is radially symmetric and non-increasing, there exists $\rho \in(0, \infty]$ such that

$$
\Omega_{2+} \subset B(0, \rho) \text { and } \Omega_{20} \subset \mathbb{R}^{d} \backslash B(0, \rho)
$$

and from (3.19) we obtain

$$
\frac{m_{2}}{m_{2}-1} g^{m_{2}-1}=\mathcal{K} * f-\frac{2 m_{1}}{M_{2}\left(d-2 m_{1}\right)}\|g\|_{m_{2}}^{m_{2}} \text { a.e. in } B(0, \rho)
$$

Then such symmetric non-increasing minimizer $(f, g) \in S_{M_{1}} \times S_{M_{2}}$ of $\mathcal{F}$ satisfies the following Euler-Lagrange equalities

$$
\left\{\begin{array}{l}
\frac{m_{1}}{m_{1}-1} f^{m_{1}-1}(x)=\mathcal{K} * g(x) \text { a.e. in } \mathbb{R}^{d},  \tag{3.20}\\
\frac{m_{2}}{m_{2}-1} g^{m_{2}-1}(x)=\mathcal{K} * f(x)-\frac{2 m_{1}}{M_{2}\left(d-2 m_{1}\right)}\|g\|_{m_{2}}^{m_{2}} \text { a.e. in } B(0, \rho) .
\end{array}\right.
$$

Step 3. The regularities of minimizer. From (3.20) ${ }_{1}$, one invokes the HLS inequality in Lemma 2.2 to see for $g \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{m_{2}}\left(\mathbb{R}^{d}\right)$ that

$$
f \in L^{p}\left(\mathbb{R}^{d}\right) \text { with } p \in\left[\frac{d\left(m_{1}-1\right)}{d-2}, \frac{d\left(m_{1}-1\right) m_{2}}{d-2 m_{2}}\right]
$$

where once more using the HLS inequality again, one concludes that

$$
\mathcal{K} * f \in L^{q}\left(\mathbb{R}^{d}\right) \text { with } q \in \begin{cases}{\left[\frac{d\left(m_{1}-1\right)}{d-2 m_{1}}, \frac{d\left(m_{1}-1\right) m_{2}}{d-2 m_{1} m_{2}}\right],} & \text { if } d>2 m_{1} m_{2} \\ {\left[\frac{d\left(m_{1}-1\right)}{d-2 m_{1}}, \infty\right),} & \text { if } d \leq 2 m_{1} m_{2}\end{cases}
$$

In particular, $\mathcal{K} * f \in L^{\frac{m_{2}}{m_{2}-1}}\left(\mathbb{R}^{d}\right)$ since $m_{1}+m_{2}=2 m_{1} / d+m_{1} m_{2} \leq 2 m_{1} m_{2} / d+$ $m_{1} m_{2}$ and

$$
\frac{m_{2}}{m_{2}-1} \in\left[\frac{d\left(m_{1}-1\right)}{d-2 m_{1}}, \frac{d\left(m_{1}-1\right) m_{2}}{\left(d-2 m_{1} m_{2}\right)_{+}}\right) .
$$

Consequently, $g^{m_{2}-1} \in L^{\frac{m_{2}}{m_{2}-1}}\left(\mathbb{R}^{d}\right)$, which excludes $\rho=\infty$ in (3.20) 2 . Hence $\rho<\infty$ and

$$
\frac{m_{2}}{m_{2}-1} g^{m_{2}-1}(x)= \begin{cases}\mathcal{K} * f(x)-\frac{2 m_{1}}{M_{2}\left(d-2 m_{1}\right)}\|g\|_{m_{2}}^{m_{2}}, & \text { if }|x|<\rho \\ 0, & \text { if }|x|>\rho\end{cases}
$$

by the monotonicity of $g$. Moreover, a bootstrap argument ensures that

$$
(f, g) \in\left(L^{\infty}\left(\mathbb{R}^{d}\right)\right)^{2}
$$

Letting $\vartheta:=f^{m_{1}-1}$ and $\varsigma:=g^{m_{2}-1}$, we readily infer from (3.20) 1 that

$$
\vartheta(x)=\frac{m_{1}-1}{m_{1}} \mathcal{K} * \varsigma^{\frac{1}{m_{2}-1}}(x) \text { a.e. in } \mathbb{R}^{d}
$$

and invoke [21, Theorem 9.9] to have $\vartheta \in W^{2, r}(B(0, \rho))$ with $r \in\left(m_{1}, \infty\right)$ and $-\Delta \vartheta=\frac{m_{1}-1}{m_{1}} \varsigma^{\frac{1}{m_{2}-1}}$ a.e. $x \in \mathbb{R}^{d}$. Furthermore, from the expression for $\varsigma$ such as

$$
\varsigma(x)=\frac{m_{2}-1}{m_{2}} \mathcal{K} * \vartheta^{\frac{1}{m_{1}-1}}(x)-\frac{2 m_{1}\left(m_{2}-1\right)}{m_{2} M_{2}\left(d-2 m_{1}\right)}\|\varsigma\|_{m_{2} /\left(m_{2}-1\right)^{\prime}}^{m_{2} /\left(m_{2}-1\right)} \quad x \in B(0, \rho)
$$

by means of the regularity of $\vartheta$ and [21, Lemma 4.2], we obtain $\varsigma \in C^{2}(B(0, \rho))$ with $-\Delta \varsigma=\frac{m_{2}-1}{m_{2}} \vartheta^{\frac{1}{m_{1}-1}}$ in $B(0, \rho)$ and [21, Lemma 4.1] ensures that $\varsigma \in C^{1}\left(\mathbb{R}^{d}\right)$. Then $\varsigma(x)=0$ if $|x|=\rho$ and $\varsigma$ is a classical solution to

$$
\left\{\begin{array}{l}
-\Delta \varsigma(x)=\frac{m_{2}-1}{m_{2}} \vartheta^{\frac{1}{m_{1}-1}}(x), \quad x \in B(0, \rho)  \tag{3.21}\\
\varsigma(x)=0, x \in \partial B(0, \rho)
\end{array}\right.
$$

With the smoothness of $\varsigma$, [21, Lemma 4.2] applies so as to assert that $\vartheta \in C^{2}\left(\mathbb{R}^{d}\right)$ and

$$
\begin{equation*}
-\Delta \vartheta(x)=\frac{m_{1}-1}{m_{1}} \zeta^{\frac{1}{m_{2}-1}}(x), \quad x \in \mathbb{R}^{d} \tag{3.22}
\end{equation*}
$$

Step 4. Contradiction. (3.21)-(3.22) consist of the Lane-Emden system (3.13). However, it has been proved that there exists no non-trivial classical solution of (3.13) if $\mathbf{m}$ is on $L_{1}$, which makes a contradiction.

Remark 3.7. Let $\mathbf{m}$ be on $L_{2}$, there exists no non-zero minimizer for $\mathcal{F}$ in $S_{M_{1}} \times S_{M_{2}}$ with $M_{1} \leq M_{1 c}$.

## 4. The global existence

This section deals with the global solvability of (1.1) in subcritical case. We first present a local existence and extensibility criterion of free energy solution to (1.1). Note that this theorem also provides simultaneous blow-up argument in Section 5.

Theorem 4.1. Let $m_{1}, m_{2}>1$. Under assumption (1.2) on the initial data $\left(u_{0}, w_{0}\right)$ with $\left\|u_{0}\right\|_{1}=M_{1},\left\|w_{0}\right\|_{1}=M_{2}$, then there exists $T_{\max } \in(0, \infty]$ and a free energy solution $(u, w)$ over $\mathbb{R}^{d} \times\left(0, T_{\max }\right)$ of (1.1) such that either $T_{\max }=\infty$ or $T_{\max }<\infty$ and

$$
\begin{equation*}
\lim _{t \rightarrow T_{\max }}\left(\|u(\cdot, t)\|_{\infty}+\|w(\cdot, t)\|_{\infty}\right)=\infty \tag{4.1}
\end{equation*}
$$

Moreover, let $\mathbf{m}$ be subcritical or critical. Then if $T_{\max }<\infty$,

$$
\begin{equation*}
\lim _{t \rightarrow T_{\max }}\|u(\cdot, t)\|_{m_{1}}=\lim _{t \rightarrow T_{\max }}\|w(\cdot, t)\|_{m_{2}}=\infty \tag{4.2}
\end{equation*}
$$

Proof. For $\left(u_{0}, w_{0}\right)$ satisfying (1.2), local existence and (4.1) can be proved by approximation arguments (similar to those in the proof of Theorem 1.1 in [43] for instance). To see (4.2), since the solution is globally solved if both $\|u\|_{m_{1}}$ and $\|w\|_{m_{2}}$ are uniform bound in subcritical or critical case due to Lemmas 2.3, 2.5 , then it is sufficient to show that the two terms $\|u\|_{m_{1}}$ and $\|w\|_{m_{2}}$ are governed by each other with some constants.

Since

$$
\begin{equation*}
\frac{1}{m_{1}-1} \int_{\mathbb{R}^{d}} u^{m_{1}}+\frac{1}{m_{2}-1} \int_{\mathbb{R}^{d}} w^{m_{2}} \leq c_{d} \mathcal{H}[u, w]+\mathcal{F}\left[u_{0}, w_{0}\right] \tag{4.3}
\end{equation*}
$$

then it needs to control the term $\mathcal{H}$ at the right side of (4.3). For $m \in(1, d / 2)$ satisfying (3.1), Lemma 3.1 yields that

$$
\begin{equation*}
|\mathcal{H}[f, g]| \leq \eta\|f\|_{m}^{m}+C \eta^{-\frac{1}{m-1}}\|g\|_{1}^{\frac{m m_{2}+2 m m_{2} / d-m-m_{2}}{(m-1)\left(m_{2}-1\right)}}\|g\|_{m_{2}}^{\frac{m_{2}-2 m m_{2} / d}{(m-1)\left(m_{2}-1\right)}} \tag{4.4}
\end{equation*}
$$

for some $f \in L^{m}\left(\mathbb{R}^{d}\right)$ and $g \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{m_{2}}\left(\mathbb{R}^{d}\right)$ with $\eta>0$. If $m_{1}<d / 2$, choosing $m=m_{1}$ in (4.4), then

$$
\begin{aligned}
& \frac{1}{m_{1}-1} \int_{\mathbb{R}^{d}} u^{m_{1}}+\frac{1}{m_{2}-1} \int_{\mathbb{R}^{d}} w^{m_{2}} \\
& \quad \leq c_{d} \eta\|u\|_{m_{1}}^{m_{1}}+c_{d} C \eta^{-\frac{1}{m_{1}-1}} M_{2}^{\frac{m_{1} m_{2}+2 m_{1} m_{2} / d-m_{1}-m_{2}}{\left(m_{1}-1\right)\left(m_{2}-1\right)}}\|w\|_{m_{2}}^{\frac{m_{2}-2 m_{1} m_{2} / d}{\left(m_{1}-1\right)\left(m_{2}-1\right)}}+\mathcal{F}\left[u_{0}, w_{0}\right] \\
& \quad \leq c_{d} \eta\|u\|_{m_{1}}^{m_{1}}+c_{d} C \eta^{-\frac{1}{m_{1}-1}}\|w\|_{m_{2}}^{m_{2}}+C
\end{aligned}
$$

by Young's inequality, since

$$
\frac{m_{2}-2 m_{1} m_{2} / d}{\left(m_{1}-1\right)\left(m_{2}-1\right)} \leq m_{2}
$$

if $m_{1} m_{2}+2 m_{1} / d \geq m_{1}+m_{2}$ holds. Taking $\eta$ small enough, we have

$$
\begin{equation*}
\|u(t)\|_{m_{1}}^{m_{1}} \leq C\|w(t)\|_{m_{2}}^{m_{2}}+C \text { for } t \in\left(0, T_{\max }\right) \tag{4.5}
\end{equation*}
$$

and if $\eta$ is sufficiently large, we see that

$$
\begin{equation*}
\|w(t)\|_{m_{2}}^{m_{2}} \leq C^{\prime}\|u(t)\|_{m_{1}}^{m_{1}}+C^{\prime} \text { for } t \in\left(0, T_{\max }\right) \tag{4.6}
\end{equation*}
$$

Therefore, (4.2) holds by (4.1), (4.5)-(4.6).
However, if $m_{1} \geq d / 2$, we pick $m \in(1, d / 2)$ such that

$$
\frac{m_{2}}{m_{2}+2 / d-1}<m<d / 2
$$

and next take interpolation inequality to find that

$$
\|u\|_{m}^{m} \leq\|u\|_{1}^{\frac{m_{1}-m}{m_{1}-1}}\|u\|_{m_{1}}^{\frac{m_{1}(m-1)}{m_{1}-1}}
$$

Upon

$$
\frac{m_{2}-2 m m_{2} / d}{(m-1)\left(m_{2}-1\right)}<m_{2}
$$

then (4.4) implies that

$$
\begin{align*}
|\mathcal{H}[u, w]| & \leq \eta\|u\|_{1}^{\frac{m_{1}-m}{m_{1}-1}}\|u\|_{m_{1}}^{\frac{m_{1}(m-1)}{m_{1}-1}}+C \eta^{-\frac{1}{m-1}}\|w\|_{1}^{\frac{m m_{2}+2 m m_{2} / d-m-m_{2}}{(m-1)\left(m_{2}-1\right)}}\|w\|_{m_{2}}^{\frac{m_{2}-2 m m_{2} / d}{(m-1)\left(m_{2}-1\right)}} \\
& =\eta M_{1}^{\frac{m_{1}-m}{m_{1}-1}}\|u\|_{m_{1}}^{\frac{m_{1}(m-1)}{m_{1}-1}}+C \eta^{-\frac{1}{m-1}} M_{2}^{\frac{m m_{2}+2 m m_{2} / d-m-m_{2}}{(m-1)\left(m_{2}-1\right)}}\|w\|_{m_{2}}^{\frac{m_{2}-2 m m_{2} / d}{(m-1)\left(m_{2}-1\right)}} \\
& \leq \eta\|u\|_{m_{1}}^{m_{1}}+\eta^{-\frac{1}{m-1}}\|w\|_{m_{2}}^{m_{2}}+C \tag{4.7}
\end{align*}
$$

with $\|u\|_{1}=M_{1}$ and $\|w\|_{1}=M_{2}$. Hence (4.5)-(4.6) are valid by picking suitable $\eta>0$. By the same token, the case $m_{1} m_{2}+2 m_{2} / d \geq m_{1}+m_{2}$ is also true for both $m_{2}<d / 2$ and $m_{2} \geq d / 2$. The proof is finished.

The global existence result in subcritical case is the subject of our next theorem.
Theorem 4.2. Let $m_{1}, m_{2}>1$. Suppose that the initial data $\left(u_{0}, w_{0}\right)$ with $\left\|u_{0}\right\|_{1}=$ $M_{1},\left\|w_{0}\right\|_{1}=M_{2}$ fulfills (1.2). Then if $\mathbf{m}$ is subcritical, (1.1) has a global free energy solution given in Definition 1.2
Remark 4.3. If $m_{1} \geq d / 2$ or $m_{2} \geq d / 2$, the conclusion in Theorem 4.2 holds for all $m_{2}>1$ or $m_{1}>1$.

Proof. In the case $m_{1} m_{2}+2 m_{1} / d>m_{1}+m_{2}$ and $m_{1}<d / 2$, since $\frac{m_{2}-2 m_{1} m_{2} / d}{\left(m_{1}-1\right)\left(m_{2}-1\right)}<$ $m_{2}$, then Lemma 3.1 warrants that

$$
\begin{aligned}
|\mathcal{H}[u, w]| & \leq \frac{1}{2 c_{d}\left(m_{1}-1\right)}\|u\|_{m_{1}}^{m_{1}}+C\|w\|_{1}^{\frac{m_{1} m_{2}+2 m_{1} m_{2} / d-m_{1}-m_{2}}{\left(m_{1}-1\right)\left(m_{2}-1\right)}}\|w\|_{m_{2}}^{\frac{m_{2}-2 m_{1} m_{2} / d}{\left(m_{1}-1\right)\left(m_{2}-1\right)}} \\
& \leq \frac{1}{2 c_{d}\left(m_{1}-1\right)}\|u\|_{m_{1}}^{m_{1}}+\frac{1}{2 c_{d}\left(m_{2}-1\right)}\|w\|_{m_{2}}^{m_{2}}+C
\end{aligned}
$$

by Young's inequality. Then substituting (4.3) into above, we have

$$
\begin{aligned}
\frac{1}{m_{1}-1} \int_{\mathbb{R}^{d}} u^{m_{1}} d x & +\frac{1}{m_{2}-1} \int_{\mathbb{R}^{d}} w^{m_{2}} d x \\
& \leq \frac{1}{2\left(m_{1}-1\right)} \int_{\mathbb{R}^{d}} u^{m_{1}} d x+\frac{1}{2\left(m_{2}-1\right)} \int_{\mathbb{R}^{d}} w^{m_{2}} d x+C
\end{aligned}
$$

As a corollary,

$$
\begin{equation*}
\|u\|_{m_{1}} \leq C \text { and }\|w\|_{m_{2}} \leq C \tag{4.8}
\end{equation*}
$$

If $m_{1} \geq \frac{d}{2}$, we recalculate (4.7) carefully and also have (4.8), in which the global existence of free energy solution is immediate from Theorem 4.1 The other case $m_{1} m_{2}+2 m_{2} / d>m_{1}+m_{2}$ is similar.

Also on the critical lines, we obtain global existence results reading as
Theorem 4.4. Let $\mathbf{m}$ be on $L_{1}$, and let $(u, w)$ be a free energy solution of (1.1) with $\left(u_{0}, w_{0}\right)$ satisfying (1.2) on $\left[0, T_{\max }\right)$ with $T_{\max }$ given in Theorem4.1 If

$$
\begin{equation*}
M_{2}<M_{2 c} \tag{4.9}
\end{equation*}
$$

then $T_{\max }=\infty$. The subcritical condition (4.9) will be replaced by $M_{1}<M_{1 c}$ on $L_{2}$. Moreover, if $\mathbf{m}$ is $\mathbf{I}$, one has $T_{\max }=\infty$ if $M_{1} M_{2}<M_{c}^{2}$.

Proof. We just infer from (1.7) and Lemma 3.3 that

$$
\begin{gathered}
\left(c_{d} C_{*}\right)^{\frac{m_{1}}{m_{1}-1}}\left(m_{1}-1\right)^{\frac{m_{1}}{m_{1}-1}} m_{1}^{-\frac{m_{1}}{m_{1}-1}}\left(M_{2 c}^{\frac{2 m_{1}}{d\left(m_{1}-1\right)}}-M_{2}^{\frac{2 m_{1}}{d\left(m_{1}-1\right)}}\right)\|w\|_{m_{2}}^{m_{2}} \\
\leq \mathcal{F}[u, w] \leq \mathcal{F}\left[u_{0}, w_{0}\right]
\end{gathered}
$$

Due to (4.9), there exists $C>0$ such that for all $t \in\left[0, T_{\max }\right)$ we have $\|w\|_{m_{2}} \leq C$ . Then the extensibility criterion in Theorem 4.1] makes sure that $T_{\max }=\infty$. The other cases can be similarly obtained.

## 5. Blow UP

Our last section concerns finite-time blow-up phenomenon when $\mathbf{m}$ is critical or super-critical. These results actually show that lines $L_{i}, i=1,2$ are optimal in view of the global existence for sub-critical case. The following second moment of solutions can be achieved in a straightforward computation.
Lemma 5.1. Let $\left(u_{0}, w_{0}\right)$ satisfy (1.2), and let $(u, w)$ be a free energy solution of (1.1) on $\left[0, T_{\max }\right)$ with $T_{\max } \in(0, \infty]$. Then

$$
\frac{d}{d t} I(t)=G(t) \text { for all } t \in\left(0, T_{\max }\right)
$$

where

$$
I(t):=\int_{\mathbb{R}^{d}}|x|^{2}(u(x, t)+w(x, t)) d x
$$

and

$$
\begin{aligned}
G(t):= & 2 d \int_{\mathbb{R}^{d}} u^{m_{1}}(x, t) d x+2 d \int_{\mathbb{R}^{d}} w^{m_{2}}(x, t) d x \\
& -2 c_{d}(d-2) \iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \frac{u(x, t) w(y, t)}{|x-y|^{d-2}} d x d y .
\end{aligned}
$$

Proof. We differentiate the second moment to see that

$$
\begin{aligned}
& \frac{d}{d t} \int_{\mathbb{R}^{d}}|x|^{2}(u(x, t)+w(x, t)) d x \\
& =\int_{\mathbb{R}^{d}}|x|^{2}\left(\Delta u^{m_{1}}-\nabla \cdot(u \nabla v)\right) d x+\int_{\mathbb{R}^{d}}|x|^{2}\left(\Delta w^{m_{2}}-\nabla \cdot(w \nabla z)\right) d x \\
& =2 d \int_{\mathbb{R}^{d}} u^{m_{1}}(x, t) d x+2 d \int_{\mathbb{R}^{d}} w^{m_{2}}(x, t) d x \\
& \quad+2 \iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}}[x \cdot \nabla \mathcal{K}(x-y)] u(x, t) w(y, t) d x d y \\
& \quad+2 \iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}}[x \cdot \nabla \mathcal{K}(x-y)] u(y, t) w(x, t) d x d y .
\end{aligned}
$$

With $\mathcal{K}(x)=c_{d} \frac{1}{|x|^{d-2}}$, we have

$$
\begin{aligned}
& 2 \iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}}[x \cdot \nabla \mathcal{K}(x-y)] u(x, t) w(y, t) d x d y \\
&=-2 c_{d}(d-2) \iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \frac{(x-y) \cdot x}{|x-y|^{d}} u(x, t) w(y, t) d x d y \\
&=-2 c_{d}(d-2) \iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \frac{|x|^{2}}{|x-y|^{d}} u(x, t) w(y, t) d x d y \\
&+2 c_{d}(d-2) \iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \frac{x \cdot y}{|x-y|^{d}} u(x, t) w(y, t) d x d y \\
&=-c_{d}(d-2) \iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \frac{|x|^{2}}{|x-y|^{d}} u(x, t) w(y, t) d x d y \\
&-c_{d}(d-2) \iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \frac{|y|^{2}}{|x-y|^{d}} u(y, t) w(x, t) d x d y \\
&+2 c_{d}(d-2) \iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \frac{x \cdot y}{|x-y|^{d}} u(x, t) w(y, t) d x d y
\end{aligned}
$$

and

$$
\begin{aligned}
2 \iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}} & {[x \cdot \nabla \mathcal{K}(x-y)] u(y, t) w(x, t) d x d y } \\
= & -c_{d}(d-2) \iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \frac{|x|^{2}}{|x-y|^{d}} u(y, t) w(x, t) d x d y \\
& -c_{d}(d-2) \iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \frac{|y|^{2}}{|x-y|^{d}} u(x, t) w(y, t) d x d y \\
& +2 c_{d}(d-2) \iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \frac{x \cdot y}{|x-y|^{d}} u(x, t) w(y, t) d x d y .
\end{aligned}
$$

Combining above equations, it follows that

$$
\begin{aligned}
\frac{d}{d t} \int_{\mathbb{R}^{d}}|x|^{2}(u(x, t)+ & w(x, t)) d x=2 d \int_{\mathbb{R}^{d}} u^{m_{1}}(x, t) d x+2 d \int_{\mathbb{R}^{d}} w^{m_{2}}(x, t) d x \\
& -c_{d}(d-2) \iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \frac{|x|^{2}+|y|^{2}}{|x-y|^{d}} u(x, t) w(y, t) d x d y \\
& -c_{d}(d-2) \iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \frac{|x|^{2}+|y|^{2}}{|x-y|^{d}} u(y, t) w(x, t) d x d y \\
& +4 c_{d}(d-2) \iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \frac{x \cdot y}{|x-y|^{d}} u(x, t) w(y, t) d x d y \\
= & 2 d \int_{\mathbb{R}^{d}} u^{m_{1}}(x, t) d x+2 d \int_{\mathbb{R}^{d}} w^{m_{2}}(x, t) d x \\
& -2 c_{d}(d-2) \iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \frac{u(x, t) w(y, t)}{|x-y|^{d-2}} d x d y
\end{aligned}
$$

which readily implies the lemma.
We construct initial data which ensures the nonnegativity of $G(0)$.
Lemma 5.2. Let $\mathbf{m}$ be critical or super-critical. There exists initial data $\left(u_{0}, w_{0}\right)$ satisfying (1.2), and fulfilling

$$
\begin{align*}
& \left(\int_{\mathbb{R}^{d}} u_{0}^{\frac{\left(m_{1}+m_{2}-m_{1} m_{2}\right) d}{2 m_{2}}} d x\right)^{\frac{2 m_{2}}{\left(m_{1}+m_{2}-m_{1} m_{2}\right) d}}\left(\int_{\mathbb{R}^{d}} w_{0}^{\frac{\left(m_{1}+m_{2}-m_{1} m_{2}\right) d}{2 m_{1}}} d x\right)^{\frac{2 m_{1}}{\left(m_{1}+m_{2}-m_{1} m_{2}\right) d}} \\
& \left(\int_{\mathbb{R}^{d}} u_{0}^{\frac{\left(m_{1}+m_{2}-m_{1} m_{2}\right) d}{2 m_{2}}} d x\right)^{\frac{2 m_{1} m_{2}}{\left(m_{1}+m_{2}-m_{1} m_{2}\right) d}}+\left(\int_{\mathbb{R}^{d}} w_{0}^{\frac{\left(m_{1}+m_{2}-m_{1} m_{2}\right) d}{2 m_{1}}} d x\right)^{\frac{2 m_{1} m_{2}}{\frac{\left(m_{1}+m_{2}-m_{1} m_{2}\right) d}{}}}  \tag{5.1}\\
& >\left\{\begin{array}{l}
N_{0}, \quad \text { if } m_{1} m_{2}+2 \max \left\{m_{1}, m_{2}\right\} / d \leq m_{1}+m_{2}<m_{1} m_{2}+2 m_{1} m_{2} / d, \\
2 N_{0}, \quad \text { if } m_{1}+m_{2} \geq m_{1} m_{2}+2 m_{1} m_{2} / d,
\end{array}\right.
\end{align*}
$$

and

$$
\begin{equation*}
G(0)<0, \tag{5.2}
\end{equation*}
$$

where

$$
N_{0}=\frac{\left(d / c_{d}\right)^{2-2 / d}}{2^{1+2 / d}(d-2)}\left(1+\frac{2 m_{1}}{\left(m_{1}+m_{2}-m_{1} m_{2}\right) d}\right)\left(1+\frac{2 m_{2}}{\left(m_{1}+m_{2}-m_{1} m_{2}\right) d}\right)
$$

and $G$ is given in Lemma 5.1
Proof. Consider the following functions having the same compact support as initial data of form

$$
\begin{align*}
& u_{0}(x)=A\left(1-\frac{|x|^{d}}{a^{d}}\right)_{+}^{\iota_{1}}, x \in \mathbb{R}^{d} \\
& w_{0}(x)=B\left(1-\frac{|x|^{d}}{a^{d}}\right)_{+}^{\iota_{2}}, x \in \mathbb{R}^{d} \tag{5.3}
\end{align*}
$$

with

$$
\begin{equation*}
\iota_{1}:=\frac{2 m_{2}}{\left(m_{1}+m_{2}-m_{1} m_{2}\right) d} \text { and } \iota_{2}:=\frac{2 m_{1}}{\left(m_{1}+m_{2}-m_{1} m_{2}\right) d^{\prime}} \tag{5.4}
\end{equation*}
$$

where $A, B>0$ denote the maximum of the supports and $a>0$ denotes the size of the supports of initial data. Such constructions in (5.3) are inspired by [44, Section 6] which deals with one-single population Keller-Segel system.

In the Case 1: $m_{1} m_{2}+2 \max \left\{m_{1}, m_{2}\right\} / d \leq m_{1}+m_{2}<m_{1} m_{2}+2 m_{1} m_{2} / d$, one has

$$
\begin{align*}
\int_{\mathbb{R}^{d}} u_{0}^{m_{1}} d x & =A^{m_{1}} \int_{\mathbb{R}^{d}}\left(1-\frac{|x|^{d}}{a^{d}}\right)_{+}^{\frac{2 m_{1} m_{2}}{\left(m_{1}+m_{2}-m_{1} m_{2}\right)^{d}}} d x \\
& =A^{m_{1}} \int_{\mathbb{R}^{d}}\left(1-\frac{|x|^{d}}{a^{d}}\right)_{+}^{\frac{2 m_{1} m_{2}}{\left(m_{1}+m_{2}-m_{1} m_{2}\right) d}}-1  \tag{5.5}\\
& \left(1-\frac{|x|^{d}}{a^{d}}\right)_{+} d x \\
& \leq A^{m_{1}} \int_{\mathbb{R}^{d}}\left(1-\frac{|x|^{d}}{a^{d}}\right)_{+} d x \\
& =c_{d} a^{d} A^{m_{1}} /(2 d)
\end{align*}
$$

and

$$
\int_{\mathbb{R}^{d}} w_{0}^{m_{2}} d x \leq c_{d} a^{d} B^{m_{2}} /(2 d)
$$

For the Case 2: $m_{1}+m_{2}>m_{1} m_{2}+2 m_{1} m_{2} / d$,

$$
\begin{aligned}
& \int_{\mathbb{R}^{d}} u_{0}^{m_{1}} d x \leq A^{m_{1}} \int_{|x|<a} 1 d x=c_{d} a^{d} A^{m_{1}} / d \\
& \int_{\mathbb{R}^{d}} w_{0}^{m_{2}} d x \leq c_{d} a^{d} B^{m_{2}} / d
\end{aligned}
$$

The coupled term can be estimated as

$$
\begin{align*}
\iint_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \frac{u_{0}(x) w_{0}(y)}{|x-y|^{d-2}} d x d y & \geq \min _{|x|,|y| \leq a}|x-y|^{-(d-2)} \int_{\mathbb{R}^{d}} u_{0}(x) d x \cdot \int_{\mathbb{R}^{d}} w_{0}(x) d x \\
& \geq a^{-(d-2)} \int_{\mathbb{R}^{d}} A\left(1-\frac{|x|^{d}}{a^{d}}\right)_{+}^{\iota_{1}} d x \cdot \int_{\mathbb{R}^{d}} B\left(1-\frac{|x|^{d}}{a^{d}}\right)_{+}^{\iota_{2}} d x \\
& =\frac{c_{d}^{2} a^{d+2}}{d^{2}\left(1+\iota_{1}\right)\left(1+\iota_{2}\right)} A B . \tag{5.6}
\end{align*}
$$

Since

$$
\begin{equation*}
G(0) \leq c_{d} a^{d} A^{m_{1}}+c_{d} a^{d} B^{m_{2}}-\frac{2 c_{d}^{3} a^{d+2}(d-2)}{d^{2}\left(1+\iota_{1}\right)\left(1+\iota_{2}\right)} A B \tag{5.7}
\end{equation*}
$$

by (5.5)-(5.6), to show (5.2), it only needs to show the right side of (5.7) is negative such that

$$
\begin{equation*}
\frac{A B}{A^{m_{1}}+B^{m_{2}}} a^{2}>N_{1} \tag{5.8}
\end{equation*}
$$

with

$$
N_{1}=\frac{d^{2}\left(1+\iota_{1}\right)\left(1+\iota_{2}\right)}{2 c_{d}^{2}(d-2)}
$$

in the Case 1, whereas the right side will be replaced by $2 N_{1}$ in the Case 2.
Since

$$
\begin{align*}
& \int_{\mathbb{R}^{d}} u_{0}^{\frac{1}{l_{1}}} d x=A^{\frac{1}{1_{1}}} \int_{\mathbb{R}^{d}}\left(1-\frac{|x|^{d}}{a^{d}}\right)_{+} d x=c_{d} a^{d} A^{\frac{1}{l_{1}}} /(2 d), \\
& \int_{\mathbb{R}^{d}} w_{0}^{\frac{1}{l_{2}}} d x=B^{\frac{1}{1_{2}}} \int_{\mathbb{R}^{d}}\left(1-\frac{|x|^{d}}{a^{d}}\right)_{+} d x=c_{d} a^{d} B^{\frac{1}{2_{2}}} /(2 d) \tag{2d}
\end{align*}
$$

implies that

$$
A=\left(\frac{2 d}{c_{d}} \int_{\mathbb{R}^{d}} u_{0}^{\frac{1}{\iota_{1}}} d x\right)^{\iota_{1}} a^{-\iota_{1} d}, \quad B=\left(\frac{2 d}{c_{d}} \int_{\mathbb{R}^{d}} w_{0}^{\frac{1}{\iota_{2}}} d x\right)^{\iota_{2}} a^{-\iota_{2} d}
$$

then (5.8) can be rewritten as

$$
\begin{aligned}
\frac{A B}{A^{m_{1}}+B^{m_{2}}} a^{2} & =\frac{\left(\frac{2 d}{c_{d}} \int_{\mathbb{R}^{d}} u_{0}^{\frac{1}{\iota_{1}}} d x\right)^{\iota_{1}}\left(\frac{2 d}{c_{d}} \int_{\mathbb{R}^{d}} w_{0}^{\frac{1}{\iota_{2}}} d x\right)^{\iota_{2}}}{\left(\frac{2 d}{c_{d}} \int_{\mathbb{R}^{d}} u_{0}^{\frac{1}{l_{1}}} d x\right)^{\iota_{1} m_{1}}+\left(\frac{2 d}{c_{d}} \int_{\mathbb{R}^{d}} w_{0}^{\frac{1}{\iota_{2}}} d x\right)^{\iota_{2} m_{2}}} \\
& =\left(\frac{2 d}{c_{d}}\right)^{\frac{2}{d}} \frac{\left(\int_{\mathbb{R}^{d}} u_{0}^{\frac{1}{\iota_{1}}} d x\right)^{\iota_{1}}\left(\int_{\mathbb{R}^{d}} w_{0}^{\frac{1}{L_{2}}} d x\right)^{\iota_{2}}}{\left(\int_{\mathbb{R}^{d}} u_{0}^{\frac{1}{l_{1}}} d x\right)^{\iota_{1} m_{1}}+\left(\int_{\mathbb{R}^{d}} w_{0}^{\frac{1}{l_{2}}} d x\right)^{\iota_{2} m_{2}}} \\
& >N_{1} \quad\left(\text { or } 2 N_{1} \text { for the Case } 2\right) .
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
& \quad\left(\int_{\mathbb{R}^{d}} u_{0}^{\frac{1}{l_{1}}} d x\right)^{\iota_{1}}\left(\int_{\mathbb{R}^{d}} w_{0}^{\frac{1}{l_{2}}} d x\right)^{\iota_{2}} \\
& \left(\int_{\mathbb{R}^{d}} u_{0}^{\frac{1}{l_{1}}} d x\right)^{\iota_{1} m_{1}}+\left(\int_{\mathbb{R}^{d}} w_{0}^{\frac{1}{l_{2}}} d x\right)^{\iota_{2} m_{2}} \\
& > \begin{cases}N_{2}, & \text { if } m_{1} m_{2}+\frac{2}{d} \max \left\{m_{1}, m_{2}\right\} \leq m_{1}+m_{2}<m_{1} m_{2}+\frac{2}{d} m_{1} m_{2}, \\
2 N_{2}, & \text { if } m_{1}+m_{2} \geq m_{1} m_{2}+\frac{2}{d} m_{1} m_{2}\end{cases}
\end{aligned}
$$

with

$$
N_{2}=\frac{\left(d / c_{d}\right)^{2-2 / d}}{2^{1+2 / d}(d-2)}\left(1+\frac{2 m_{1}}{\left(m_{1}+m_{2}-m_{1} m_{2}\right) d}\right)\left(1+\frac{2 m_{2}}{\left(m_{1}+m_{2}-m_{1} m_{2}\right) d}\right)
$$

which yields $G(0)<0$ with $N_{0}=N_{2}$.
The blow-up results state that
Theorem 5.3. Let $\mathbf{m}$ be critical or super-critical. Then one can find some initial data $\left(u_{0}, w_{0}\right)$ satisfying (1.2) such that free energy solution $(u, w)$ of (1.1) with $\left.(u, w)\right|_{t=0}=$ $\left(u_{0}, w_{0}\right)$ blows up in finite time.

Proof. For a given initial data $\left(u_{0}, w_{0}\right)$ in (5.3) satisfying (5.1), then $G(0)<0$ from Lemma5.2. By the continuity argument, there exists $T^{*}>0$ such that

$$
G(t)<G(0) / 2 \text { for all } t \in\left[0, T^{*}\right]
$$

where from Lemma 5.1, one obtains $\frac{d}{d t} I(t)<G(0) / 2$ for all $t \in\left[0, T^{*}\right]$. Integrating by parts, it follows that

$$
\begin{equation*}
I\left(T^{*}\right)<I(0)+G(0) T^{*} / 2 . \tag{5.9}
\end{equation*}
$$

As

$$
\begin{align*}
I(0) & =\int_{\mathbb{R}^{d}}|x|^{2}\left(A\left(1-\frac{|x|^{d}}{a^{d}}\right)_{+}^{\iota_{1}}+B\left(1-\frac{|x|^{d}}{a^{d}}\right)_{+}^{\iota_{2}}\right) d x \\
& =A \int_{|x| \leq a}|x|^{2}\left(1-\frac{|x|^{d}}{a^{d}}\right)^{\iota_{1}} d x+B \int_{|x| \leq a}|x|^{2}\left(1-\frac{|x|^{d}}{a^{d}}\right)^{\iota_{2}} d x \\
& =c_{d} A \int_{0}^{a}\left(1-\frac{r^{d}}{a^{d}}\right)^{\iota_{1}} r^{d+1} d r+c_{d} B \int_{0}^{a}\left(1-\frac{r^{d}}{a^{d}}\right)^{\iota_{2}} r^{d+1} d r  \tag{5.10}\\
& =\left(c_{d} a^{d+2} A\right) / d \int_{0}^{1}(1-r)^{\iota_{1}} r^{2 / d} d r+\left(c_{d} a^{d+2} B\right) / d \int_{0}^{1}(1-r)^{\iota_{2}} r^{2 / d} d r \\
& =\left(c_{d} a^{d+2} A N_{3}\right) / d+\left(c_{d} a^{d+2} B N_{4}\right) / d
\end{align*}
$$

with $\iota_{1}, \iota_{2}$ given in (5.4) and

$$
N_{3}:=\int_{0}^{1}(1-r)^{l_{1}} r^{2 / d} d r<\infty \text { and } N_{4}:=\int_{0}^{1}(1-r)^{\iota_{2}} r^{2 / d} d r<\infty,
$$

then inserting (5.7) and (5.10) into (5.9), the right side of (5.9) should be negative if we may fix small $a>0$ such that

$$
\begin{aligned}
& \frac{T^{*}}{2} \cdot\left[\frac{2 c_{d}^{3} a^{d+2}(d-2)}{d^{2}\left(1+\iota_{1}\right)\left(1+\iota_{2}\right)} A B-c_{d} a^{d} A^{m_{1}}-c_{d} a^{d} B^{m_{2}}\right] \\
& \quad \geq\left(c_{d} a^{d+2} A N_{3}\right) / d+\left(c_{d} a^{d+2} B N_{4}\right) / d
\end{aligned}
$$

More precisely, if

$$
\begin{aligned}
& \frac{d T^{*}}{2} \cdot\left[\frac{2^{1+2 / d}(d-2)}{\left(1+\iota_{1}\right)\left(1+\iota_{2}\right)}\left(\frac{c_{d}}{d}\right)^{2-2 / d} \cdot\left(\int_{\mathbb{R}^{d}} u_{0}^{\frac{1}{\iota_{1}}} d x\right)^{\iota_{1}}\left(\int_{\mathbb{R}^{d}} w_{0}^{\frac{1}{2_{2}}} d x\right)^{\iota_{2}}\right. \\
& \left.-\left(\int_{\mathbb{R}^{d}} u_{0}^{\frac{1}{\iota_{1}}} d x\right)^{m_{1} \iota_{1}}-\left(\int_{\mathbb{R}^{d}} w_{0}^{\frac{1}{l_{2}}} d x\right)^{m_{2} \iota_{2}}\right] \\
& \geq\left(\frac{2 d}{c_{d}}\right)^{\left(1-m_{1}\right) \iota_{1}}\left(\int_{\mathbb{R}^{d}} u_{0}^{\frac{1}{\iota_{1}}} d x\right)^{\iota_{1}} a^{d \iota_{2}} N_{3}+\left(\frac{2 d}{c_{d}}\right)^{\left(1-m_{2}\right) \iota_{2}}\left(\int_{\mathbb{R}^{d}} w_{0}^{\frac{1}{\iota_{2}}} d x\right)^{\iota_{2}} a^{d \iota_{1}} N_{4},
\end{aligned}
$$

this leads to a contradiction after time $T^{*}$ since $I(t)$ is always nonnegative for all $t>0$. Hence the solutions blow up in finite time.

If $\mathbf{m}$ is $\mathbf{I}$, Theorem 5.3 shows that the blow up condition (5.1) can be written as

$$
\begin{equation*}
\frac{M_{1} M_{2}}{M_{1}^{m_{c}}+M_{2}^{m_{c}}}>\frac{1}{2(d-2)} \cdot\left(\frac{2 d}{c_{d}}\right)^{m_{c}}, \tag{5.11}
\end{equation*}
$$

since

$$
\frac{d}{d t} I(t)=G(t)=2(d-2) \mathcal{F}[u(t), w(t)] \leq 2(d-2) \mathcal{F}\left[u_{0}, w_{0}\right]=G(0)<0
$$

if (5.11) holds, then the second moment will be negative after some time and it contradicts the non-negativity of $u$ and $w$.

We improve blow-up arguments if $\mathbf{m}$ is $\mathbf{I}$ by using a different method and summarize the blows up results on the lines $L_{1}, L_{2}$ and intersection point $\mathbf{I}$ as
Theorem 5.4. Let $\mathbf{m}$ be critical. Suppose that $(u, w)$ is a free energy solution of (1.1) with $\left\|u_{0}\right\|_{1}=M_{1},\left\|w_{0}\right\|_{1}=M_{2}$ fulfilling (1.2).

If $\mathbf{m}$ is on $L_{1}$, for sufficiently small size of the supports of $\left(u_{0}, w_{0}\right)$ one asserts that blow up happens if

$$
\frac{\left(\int_{\mathbb{R}^{d}} u_{0}^{m_{1} / m_{2}} d x\right)^{m_{2} / m_{1}}\left(\int_{\mathbb{R}^{d}} w_{0} d x\right)}{\left(\int_{\mathbb{R}^{d}} u_{0}^{m_{1} / m_{2}} d x\right)^{m_{2}}+\left(\int_{\mathbb{R}^{d}} w_{0} d x\right)^{m_{2}}}>N_{0}
$$

with $N_{0}$ given in Lemma 5.2
If $\mathbf{m}$ is on $L_{2}$, for sufficiently small size of the supports of $\left(u_{0}, w_{0}\right)$ blow-up solution can be constructed if

$$
\frac{\left(\int_{\mathbb{R}^{d}} u_{0} d x\right)\left(\int_{\mathbb{R}^{d}} w_{0}^{m_{2} / m_{1}} d x\right)^{m_{1} / m_{2}}}{\left(\int_{\mathbb{R}^{d}} u_{0} d x\right)^{m_{1}}+\left(\int_{\mathbb{R}^{d}} w_{0}^{m_{2} / m_{1}} d x\right)^{m_{1}}}>N_{0} .
$$

If $\mathbf{m}$ is $\mathbf{I}$, blow up occurs if

$$
M_{1} M_{2} /\left(M_{1}^{m_{c}}+M_{2}^{m_{c}}\right)>M_{c}^{2 / d} / 2
$$

Finally, let $(u, w)$ blow up in finite time $T_{\max }$. Then $T_{\max }<\infty$ implies that

$$
\lim _{t \rightarrow T_{\max }}\|u\|_{m_{1}}=\lim _{t \rightarrow T_{\max }}\|w\|_{m_{2}}=\infty
$$

Proof. The asserted blow-up conditions on the lines $L_{1}$ and $L_{2}$ just follow from Lemma5.2 and Theorem5.3. If $\mathbf{m}$ is I, note that for any $M_{1}^{*}>0$ and $M_{2}^{*}>0$ such that

$$
\begin{equation*}
M_{1}^{*} M_{2}^{*} /\left(M_{1}^{* m_{c}}+M_{2}^{* m_{c}}\right)=M_{c}^{2 / d} / 2 \tag{5.12}
\end{equation*}
$$

there exists nonnegative function $\left(u^{*}, w^{*}\right)$ with $\left\|u^{*}\right\|_{1}=M_{1}^{*},\left\|w^{*}\right\|_{1}=M_{2}^{*}$ fulfilling $\mathcal{F}\left[u^{*}, w^{*}\right]=0$.

This can be seen by the fact that $C_{c}$ in (3.6) is

$$
C_{c}=\sup _{f \neq 0}\left\{\frac{\mathcal{H}[f, f]}{\|f\|_{1}^{2 / d}\|f\|_{m_{c}}^{m_{c}}}, f \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{m_{c}}\left(\mathbb{R}^{d}\right)\right\}
$$

from Theorem 3.4. From [6, Proposition 3.3], for any $M_{1}^{*}>0$ there exists nonnegative, radially symmetric and non-increasing function $u^{*} \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{m_{c}}\left(\mathbb{R}^{d}\right)$ with $\left\|u^{*}\right\|_{1}=M_{1}^{*}$ such that

$$
\begin{equation*}
\left\|u^{*}\right\|_{m_{c}}^{m_{c}}=C_{c}^{-1}\left\|u^{*}\right\|_{1}^{-2 / d} \mathcal{H}\left[u^{*}, u^{*}\right] . \tag{5.13}
\end{equation*}
$$

Define $w^{*}=M_{2}^{*} / M_{1}^{*} u^{*}$. Then $w^{*} \in L^{1}\left(\mathbb{R}^{d}\right) \cap L^{m_{c}}\left(\mathbb{R}^{d}\right)$ with $\left\|w^{*}\right\|_{1}=M_{2}^{*}$ and

$$
\mathcal{F}\left[u^{*}, w^{*}\right]=0
$$

by (5.12) and the definition of $M_{c}$. Then

$$
c_{d} \mathcal{H}\left[u^{*}, w^{*}\right]=c_{d} M_{2}^{*} / M_{1}^{*} \mathcal{H}\left[u^{*}, u^{*}\right]=\frac{1}{m_{c}-1}\left(1+\left(\frac{M_{2}^{*}}{M_{1}^{*}}\right)^{m_{c}}\right)\left\|u^{*}\right\|_{m_{c}}^{m_{c}} .
$$

Given $u_{0}=\frac{M_{1}}{M_{1}^{*}} u^{*}$ and $w_{0}=\frac{M_{2}}{M_{2}^{*}} w^{*}$ with $\left\|u_{0}\right\|_{1}=M_{1}$ and $\left\|w_{0}\right\|_{1}=M_{2}$, then

$$
\begin{aligned}
\mathcal{F}\left[u_{0}, w_{0}\right] & =\frac{1}{m_{c}-1}\left\|u_{0}\right\|_{m_{c}}^{m_{c}}+\frac{1}{m_{c}-1}\left\|w_{0}\right\|_{m_{c}}^{m_{c}}-c_{d} \mathcal{H}\left[u_{0}, w_{0}\right] \\
& =\frac{1}{m_{c}-1}\left[\left(\frac{M_{1}}{M_{1}^{*}}\right)^{m_{c}}+\left(\frac{M_{2}}{M_{1}^{*}}\right)^{m_{c}}-\frac{M_{1} M_{2}}{M_{1}^{*} M_{2}^{*}}\left(1+\left(\frac{M_{2}^{*}}{M_{1}^{*}}\right)^{m_{c}}\right)\right]\left\|u^{*}\right\|_{m_{c}}^{m_{c}} \\
& <0,
\end{aligned}
$$

since

$$
M_{1} M_{2} /\left(M_{1}^{m_{c}}+M_{2}^{m_{c}}\right)>M_{1}^{*} M_{2}^{*} /\left(M_{1}^{* m_{c}}+M_{2}^{* m_{c}}\right)=M_{c}^{2 / d} / 2 .
$$

If $(u, w)$ is corresponding free energy solution with the initial data $\left(u_{0}, w_{0}\right)$, then

$$
\mathcal{F}[u(t), w(t)] \leq \mathcal{F}\left[u_{0}, w_{0}\right]<0, \quad t>0
$$

by the decreasing property of $\mathcal{F}$. From Lemma 5.1, it follows that blow up occurs.
To see the simultaneous blow-up phenomenon, from extensibility criterion in Theorem 4.1 we have

$$
C\|w(t)\|_{m_{2}}^{m_{2}}+C \leq\|u(t)\|_{m_{1}}^{m_{1}} \leq C^{\prime}\|w(t)\|_{m_{2}}^{m_{2}}+C^{\prime} \text { for } t \in\left(0, T_{\max }\right)
$$

with some $C>0$ and $C^{\prime}>0$ if $\mathbf{m}$ is critical. Then all assertions have been proved.

Acknowledgment. JAC was supported the Advanced Grant Nonlocal-CPD (Nonlocal PDEs for Complex Particle Dynamics: Phase Transitions, Patterns and Synchronization) of the European Research Council Executive Agency (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 883363). JAC was also partially supported by the EPSRC grant number EP/P031587/1. JAC acknowledges support through the Changjiang Visiting Professorship Scheme of the Chinese Ministry of Education. KL is partially supported by NSFC (Grant No. 11601516) and by Sichuan Science and Technology Program (Grant No. 2020YJ0060).

## References

[1] L. Ambrosio, N. Gigli, G. Savaré, Gradient flows in metric spaces and in the space of probability measures, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2005.
[2] J. Bedrossian, Global minimizers for free energies of subcritical aggregation equations with degenerate diffusion, Appl. Math. Lett. 24 (2011) 1927-1932.
[3] N. Bellomo, A. Bellouquid, Y. Tao, M. Winkler, Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues, Math. Models Methods Appl. Sci. 25 (2015) 16631763.
[4] S. Bian, J.-G. Liu, Dynamic and steady states for multi-dimensional Keller-Segel model with diffusion exponent $m>0$, Comm. Math. Phys. 323 (2013) 1017-1070.
[5] A. Blanchet, E.A. Carlen, J.A. Carrillo, Functional inequalities, thick tails and asymptotics for the critical mass Patlak-Keller-Segel model, J. Funct. Anal. 262 (2012) 2142-2230.
[6] A. Blanchet, J. A. Carrillo, P. Laurencot, Critical mass for a Patlak-Keller-Segel model with degenerate diffusion in higher dimensions, Calc. Var. Partial Differential Equations 35 (2009) 133-168.
[7] A. Blanchet, J.A. Carrillo, N. Masmoudi, Infinite time aggregation for the critical two-dimensional Patlak-Keller-Segel model, Comm. Pure Appl. Math. 61 (2008) 1449-1481.
[8] A. Blanchet, J. Dolbeault, B. Perthame, Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions, Electron. J. Differential Equations 44 (2006), 32 pp. (electronic).
[9] V. Calvez, J.A. Carrillo, Volume effects in the Keller-Segel model: energy estimates preventing blow-up, J. Math. Pures Appl. 86 (2006) 155-175.
[10] V. Calvez, J. A. Carrillo, F. Hoffmann, Equilibria of homogeneous functionals in the faircompetition regime, Nonlinear Anal. 159 (2017) 85-128.
[11] V. Calvez, L. Corrias, M.A. Ebde, Blow-up, concentration phenomenon and global existence for the Keller-Segel model in high dimension, Comm. Partial Differential Equations 37 (2012) 561-584.
[12] J.A. Carrillo, D. Castorina, B. Volzone, Ground states for diffusion dominated free energies with logarithmic interaction, SIAM J. Math. Anal. 47 (2015) 1-25.
[13] J.A. Carrillo, K. Craig, Y. Yao, Aggregation-diffusion equations: dynamics, asymptotics, and singular limits. Active particles, Vol. 2., 65-108, Model. Simul. Sci. Eng. Technol., Birkhäuser/Springer, Cham, 2019.
[14] J.A. Carrillo, S. Hittmeir, B. Volzone, Y. Yao, Nonlinear aggregation-diffusion equations: radial symmetry and long time asymptotics, Invent. Math. 218 (2019) 889-977.
[15] J.A. Carrillo, F. Hoffmann, E. Mainini, B. Volzone, Ground states in the diffusion-dominated regime, Calc. Var. Partial Differential Equations 57 (2018) Art. 127, 28 pp.
[16] L. Chen, J.H. Wang, Exact criterion for global existence and blow up to a degenerate Keller-Segel system, Doc. Math. 19 (2014) 103-120.
[17] L. Chen, J.-G. Liu, J. Wang, Multidimensional degenerate Keller-Segel system with critical diffusion exponent $2 n /(n+2)$, SIAM J. Math. Anal. 44 (2012) 1077-1102.
[18] L. Corrias, B. Perthame, H. Zaag, Global solutions of some chemotaxis and angiogenesis systems in high space dimensions, Milan J. Math. 72 (2004) 1-28.
[19] J. Dolbeault, B. Perthame, Optimal critical mass in the two dimensional Keller-Segel model in $\mathbb{R}^{2}$, C. R. Math. Acad. Sci. Paris 339 (2004) 611-616.
[20] E. Espejo, K. Vilches, C. Conca, A simultaneous blow-up problem arising in tumor modeling, J. Math. Biol. 79 (2019) 1357-1399.
[21] D. Gilbarg, N.S. Trudinger, Elliptic partial differential equations of second order. In: Grundlehren derMathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 224, 2nd edn. Springer, Berlin (1983).
[22] L. Hong, J.H. Wang, H. Yu, Y. Zhang, Critical mass for a two-species chemotaxis model with two chemicals in $\mathbb{R}^{2}$, Nonlinearity 32 (2019) 4762-4778.
[23] D. Horstmann, From 1970 until present: The Keller-Segel model in chemotaxis and its consequences. I, Jahresber. Deutsch. Math.-Verein. 105 (2003) 103-165.
[24] W. Jäger, S. Luckhaus, On explosions of solutions to a system of partial differential equations modelling chemotaxis, Trans. Amer. Math. Soc. 329 (1992) 819-824.
[25] R. Jordan, D. Kinderlehrer, F. Otto, The variational formulation of the Fokker Planck equation, SIAM J. Math. Anal. 29 (1998) 1-17.
[26] D. Karmakar, G. Wolansky, On Patlak-Keller-Segel system for several populations: A gradient flow approach, J. Differential Equations 267 (2019) 7483-7520.
[27] D. Karmakar, G. Wolansky, On the critical mass Patlak-Keller-Segel for multi-species populations: Global existence and infinite time aggregation, arXiv:2004.10132(2020).
[28] E.F. Keller, L.A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theor. Biol. 26 (1970) 399-415.
[29] I. Kim, Y. Yao, The Patlak-Keller-Segel model and its variations: properties of solutions via maximum principle, SIAM J. Math. Anal. 44 (2012) 568-602.
[30] H. Knútsdóttir, E. Pálsson, L. Edelstein-Keshet, Mathematical model of macrophage-facilitated breast cancer cells invasion, J. Theor. Biol. 357 (2014) 184-199.
[31] E.H. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, Ann. Math. 118 (1983) 349-374.
[32] E.H. Lieb, M. Loss, Analysis. In: Graduate Studies in Mathematics, vol. 14, 2nd edn. American Mathematical Society, Providence (2001).
[33] E. H. Lieb, H.-T. Yau, The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics, Comm. Math. Phys. 112 (1987) 147-174.
[34] K. Lin, T. Xiang, On global solutions and blow-up for a short-ranged chemical signaling loop, J. Nonlinear Sci. 29 (2019) 551-591.
[35] K. Lin, T. Xiang, On boundedness, blow-up and convergence in a two-species and two-stimuli chemotaxis system with/without loop, Calc. Var. Partial Differential Equations 59 (2020), doi: https:/ /doi.org/10.1007/s00526-020-01777-7.
[36] P. L. Lions, The concentration-compactness principle in calculus of variations. The locally compact case, Part 1, Ann. Inst. H. Poincaré 1 (1984) 109-145.
[37] E. Mitidieri, Nonexistence of positive solutions of semilinear elliptic systems in $\mathbb{R}^{d}$, Differ. Integral Equations 9 (1996) 465-479.
[38] K. Painter, T. Hillen, Volume-filling and quorum-sensing in models for chemosensitive movement, Can. Appl. Math. Q. 10 (2002) 501-543.
[39] B. Perthame, Transport Equation in Biology, Frontiers in Mathematics, Birkháuser, 2007.
[40] J. Serrin, H. Zou, Non-existence of positive solutions of the Lane-Emden system, Differ. Integral Equations 9 (1996) 635-653.
[41] J. Serrin, H. Zou, Existence of positive solutions of the Lane-Emden system, Atti Semi. Mat. Fis. Univ. Modena 46 (1998) 369-380.
[42] I. Shafrir, G. Wolansky, Moser-Trudinger and logarithmic HLS inequalities for systems, J. Eur. Math. Soc. 7 (2005) 413-448.
[43] Y. Sugiyama, Global existence in sub-critical cases and finite time blow-up in super-critical cases to degenerate Keller-Segel systems, Differ. Integral Equations 19 (2006) 864-876.
[44] Y. Sugiyama, Application of the best constant of the Sobolev inequality to degenerate Keller-Segel models, Advances in Differential Equations 12 (2007) 121-144.
[45] Y. Sugiyama, H. Kunii, Global existence and decay properties for a degenerate Keller-Segel model with a power factor in drift term, J. Differential Equations 227 (2006) 333-364.
[46] Y. Tao, M. Winkler, Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity, J. Differential Equations 252 (2012) 692-715.
[47] Y. Tao, M. Winkler, Boundedness vs. blow-up in a two-species chemotaxis system with two chemicals, Discrete Contin. Dyn. Syst. Ser. B 20 (2015) 3165-3183.
[48] H. Yu, W. Wang, S. Zheng, Criteria on global boundedness versus finite time blow-up to a twospecies chemotaxis system with two chemicals, Nonlinearity 31 (2018) 502-514.

Mathematical Institute, University of Oxford, Oxford, OX2 6GG, UK
Email address: carrillo@maths.ox.ac.uk
School of Economics and Mathematics, Southwestern University of Economics and Finance, Chengdu, 610074 SICHU China

Email address: linke@swufe.edu.cn


[^0]:    2010 Mathematics Subject Classification. 35K65, 92C17, 35J20, 35A01, 35B44
    Key words and phrases. Degenerate parabolic system, chemotaxis, variational methods, global existence, blow up.

